NAME: <u>KEY</u>

MATH 127 TEST 2 SAMPLE

NOTE: The actual exam will only have 11 questions. The different parts of each question (parts A, B) are variations. Know how to do all the variations on this exam.

1A.) (6 points) Establish the identity: $\dfrac{\cos x - 1}{\sin x} + \dfrac{\sin x}{1+\cos x} = 0$

$$\frac{\cos x - 1}{\sin x}\cdot\left(\frac{1+\cos x}{1+\cos x}\right) + \frac{\sin x}{1+\cos x}\cdot\left(\frac{\sin x}{\sin x}\right) = 0$$

$$\frac{\cos x + \cos^2 x - 1 - \cos x + \sin^2 x}{\sin x(1+\cos x)} = 0$$

$$\frac{\cos^2 x + \sin^2 x - 1}{\sin x(1+\cos x)} = 0$$

$$\frac{1-1}{\sin x(1+\cos x)} = 0$$

$$0 = 0$$

1B.) (6 points) Establish the identity: $\dfrac{1-\sin x}{\cos x} + \dfrac{\cos x}{1-\sin x} = 2\sec x$

$$\frac{1-\sin x}{\cos x}\cdot\left(\frac{1-\sin x}{1-\sin x}\right) + \frac{\cos x}{1-\sin x}\cdot\left(\frac{\cos x}{\cos x}\right) = 2\cdot\frac{1}{\cos x}$$

$$\frac{1-2\sin x + \sin^2 x + \cos^2 x}{\cos x(1-\sin x)} = \frac{2}{\cos x}$$

$$\frac{1-2\sin x + 1}{\cos x(1-\sin x)} = \frac{2}{\cos x}$$

$$\frac{2-2\sin x}{\cos x(1-\sin x)} = \frac{2}{\cos x}$$

$$\frac{2(1-\sin x)}{\cos x(1-\sin x)} = \frac{2}{\cos x}$$

$$\frac{2}{\cos x} = \frac{2}{\cos x}$$

2A.) (6 pts) Establish the identity: $\dfrac{\cos\theta}{\sec\theta-\tan\theta}=1+\sin\theta$

$$\frac{\cos\theta}{\frac{1}{\cos}-\frac{\sin\theta}{\cos\theta}}=1+\sin\theta$$

$$\frac{\frac{\cos\theta}{1}}{\frac{1-\sin\theta}{\cos}}=1+\sin\theta$$

$$\frac{\cos\theta}{1}\cdot\frac{\cos\theta}{1-\sin\theta}=1+\sin\theta$$

$$\frac{\cos^2\theta}{1-\sin\theta}=1+\sin\theta$$

$$\frac{1-\sin^2\theta}{1-\sin\theta}=1+\sin\theta$$

$$\frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta}=1+\sin\theta$$

$$1+\sin\theta=1+\sin\theta$$

2B.) (6 pts) Establish the identity: $\dfrac{\sec\theta-\cos\theta}{\sec\theta}=\sin^2\theta$

$$\frac{\frac{1}{\cos\theta}-\cos\theta}{\frac{1}{\cos\theta}}=\sin^2\theta$$

$$\frac{\frac{1-\cos^2\theta}{\cos\theta}}{\frac{1}{\cos\theta}}=\sin^2\theta$$

$$\frac{1-\cos^2\theta}{\cos\theta}\cdot\frac{\cos\theta}{1}=\sin^2\theta$$

$$1-\cos^2\theta=\sin^2\theta$$

$$\sin^2\theta=\sin^2\theta$$

3A.) (4 pts) Establish the identity by using sum or difference formulas:
$\sin(180^\circ - x) + \cos(x + 90^\circ) = 0$

$$\sin 180^\circ \cos x - \cos 180^\circ \sin x + \cos x \cos 90^\circ - \sin x \sin 90^\circ = 0$$

$$(0)\cos x - (-1)\sin x + \cos x(0) - \sin x(1) = 0$$

$$\sin x - \sin x = 0$$

$$0 = 0$$

3B.) (4 pts) Establish the identity by using sum or difference formulas:
$$\frac{\cos\left(x - \frac{\pi}{2}\right)}{\sin\left(x + \frac{\pi}{2}\right)} = \tan x$$

$$\frac{\cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2}}{\sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}} = \frac{\sin x}{\cos x}$$

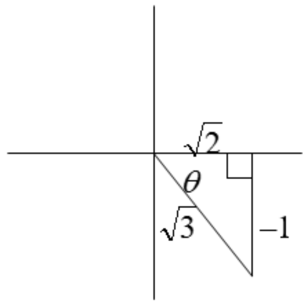
$$\frac{\cos x(0) + \sin x(1)}{\sin x(0) + \cos x(1)} = \frac{\sin x}{\cos x}$$

$$\frac{\sin x}{\cos x} = \frac{\sin x}{\cos x}$$

4A.) (10 points) Find the exact values given:

$$\cot\theta = -\sqrt{2} \text{ and } \cos\theta > 0.$$

We are given than the cotangent is negative and cosine is positive. This only occurs in the 4th quadrant. Label the triangle and use Pythagorean theorem to find the missing side.



$\sin\theta = -\dfrac{\sqrt{3}}{3}$ $\quad\csc\theta = -\sqrt{3}$

$\cos\theta = \dfrac{\sqrt{6}}{3}$ $\quad\sec\theta = \dfrac{\sqrt{6}}{2}$

$\tan\theta = -\dfrac{\sqrt{2}}{2}$ $\quad\sin(2\theta) = -\dfrac{2\sqrt{2}}{3}$

$\cos(2\theta) = \dfrac{1}{3}$ $\quad\tan(2\theta) = -2\sqrt{2}$

$\sin\dfrac{\theta}{2} = \sqrt{\dfrac{3-\sqrt{6}}{6}}$ $\quad\tan\dfrac{\theta}{2} = -\dfrac{\sqrt{3}}{3+\sqrt{6}}$

Now we can get our first 5 trigonometric functions by reading off our triangle:

$\sin\theta = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$ $\quad\csc\theta = -\sqrt{3}$ $\quad\cos\theta = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$ $\quad\sec\theta = \dfrac{3}{\sqrt{6}} = \dfrac{3\sqrt{6}}{6} = \dfrac{\sqrt{6}}{2}$

$\tan\theta = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$\sin(2\theta) = 2\sin\theta\cos\theta$.

$$\sin(2\theta) = 2\left(-\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) = -\frac{2\sqrt{18}}{9} = -\frac{6\sqrt{2}}{9} = -\frac{2\sqrt{2}}{3}.$$

$\cos(2\theta) = 2\cos^2\theta - 1$

$$\cos(2\theta) = 2\left(\frac{\sqrt{6}}{3}\right)^2 - 1 = 2\left(\frac{6}{9}\right) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

$$\tan(2\theta) = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}.$$

$270^\circ \le \theta \le 360^\circ$, so $135^\circ \le \dfrac{\theta}{2} \le 180^\circ$. $\dfrac{\theta}{2}$ is in the second quadrant, so sine of $\dfrac{\theta}{2}$ should be positive

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}, \text{ so } \sin\frac{\theta}{2} = \sqrt{\frac{1-\left(\frac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\frac{\frac{3-\sqrt{6}}{3}}{2}} = \sqrt{\frac{3-\sqrt{6}}{6}}.$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}, \text{ so } \tan\frac{\theta}{2} = \frac{-\frac{\sqrt{3}}{3}}{1+\left(\frac{\sqrt{6}}{3}\right)} = \frac{-\frac{\sqrt{3}}{3}}{\frac{3+\sqrt{6}}{3}} = -\frac{\sqrt{3}}{3}\cdot\frac{3}{3+\sqrt{6}} = -\frac{\sqrt{3}}{3+\sqrt{6}}$$

This answer is okay.

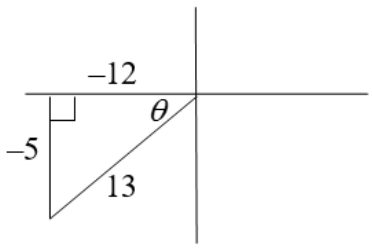
4B.) (10 points) Find the exact values given:

$\tan\theta = \frac{5}{12}$ and $180^\circ \le \theta \le 270^\circ$.

We are given that the triangle is in the third quadrant.

Remember that both the x and y values should be negative.

Label the sides and use the Pythagorean Theorem.



$\sin\theta = -\frac{5}{13}$ $\csc\theta = -\frac{13}{5}$

$\cos\theta = -\frac{12}{13}$ $\sec\theta = -\frac{13}{12}$

$\cot\theta = \frac{12}{5}$ $\sin(2\theta) = \frac{120}{169}$

$\cos(2\theta) = \frac{119}{169}$ $\tan(2\theta) = \frac{120}{119}$

$\cos\frac{\theta}{2} = -\sqrt{\frac{1}{26}}$ $\tan\frac{\theta}{2} = -5$

Now we can get our first 5 trigonometric functions by reading off our triangle:

$\sin\theta = -\frac{5}{13}$ $\csc\theta = -\frac{13}{5}$ $\cos\theta = -\frac{12}{13}$ $\sec\theta = -\frac{13}{12}$

$\cot\theta = \frac{12}{5}$

$\sin(2\theta) = 2\sin\theta\cos\theta$.

$\sin(2\theta) = 2\left(-\frac{5}{13}\right)\left(-\frac{12}{13}\right) = \frac{120}{169}$.

$\cos(2\theta) = 1 - 2\sin^2\theta$

$\cos(2\theta) = 1 - 2\left(-\frac{5}{13}\right)^2 = 1 - 2\left(\frac{25}{169}\right) = 1 - \frac{50}{169} = \frac{119}{169}$

$$\tan(2\theta) = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{120}{169}}{\frac{119}{169}} = \frac{120}{119}.$$

$180^\circ \le \theta \le 270^\circ$, so $90^\circ \le \frac{\theta}{2} \le 135^\circ$. $\frac{\theta}{2}$ is in the second quadrant, so cosine of $\frac{\theta}{2}$ should be negative

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}, \text{ so } \cos\frac{\theta}{2} = -\sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}} = -\sqrt{\frac{\frac{1}{13}}{2}} = -\sqrt{\frac{1}{26}}.$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}, \text{ so } \tan\frac{\theta}{2} = \frac{-\frac{5}{13}}{1+\left(-\frac{12}{13}\right)} = \frac{-\frac{5}{13}}{\frac{1}{13}} = -5$$

5A.) (5 pts) Establish the identity by using double angle formulas:

$$\frac{\sin 2x - \sin x}{\cos 2x + \cos x} = \frac{\sin x}{\cos x + 1}$$

For this problem we will first substitute the double angle identities. For cos(2*x*), we want to choose the specific identity that has only one cosine since we see there is already another cosine next to it.

$$\frac{(2\sin x\cos x) - \sin x}{(2\cos^2 x - 1) + \cos x} = \frac{\sin x}{\cos x + 1}$$

Next, we want to rearrange the denominator.

$$\frac{2\sin x\cos x - \sin x}{2\cos^2 x + \cos x - 1} = \frac{\sin x}{\cos x + 1}$$

Now we will factor the top and bottom separately.

$$\frac{\sin x(2\cos x - 1)}{(2\cos x - 1)(\cos x + 1)} = \frac{\sin x}{\cos x + 1}$$

$$\frac{\sin x}{\cos x + 1} = \frac{\sin x}{\cos x + 1}$$

Both sides are equal, so the identity is established.

5B.) (5 pts) Establish the identity by using double angle formulas:

$$\sin x - \cos(2x) = (2\sin x - 1)(\sin x + 1)$$

First, let's distribute the right side using the FOIL method:

$$\sin x - \cos(2x) = 2\sin^2 x + 2\sin x - \sin x - 1$$

Simplify.

$$\sin x - \cos(2x) = 2\sin^2 x + \sin x - 1$$

Now we will use the identity for cos(2*x*) that has only sine since there is another sine next to it:

$$\sin x - (1 - 2\sin^2 x) = 2\sin^2 x + \sin x - 1$$

Distribute the minus sign.

$$\sin x - 1 + 2\sin^2 x = 2\sin^2 x + \sin x - 1$$

Now rearrange.

$$2\sin^2 x + \sin x - 1 = 2\sin^2 x + \sin x - 1$$

Both identities are equal, so we are done.

6A.) (4 pts) Find the exact value of $\cos\left(\frac{5\pi}{24}\right)\sin\left(\frac{\pi}{24}\right)$ using a product-to-sum formula.

6A. $\frac{\sqrt{2}-1}{4}$

Use $\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$

$$= \frac{1}{2}\left[\sin\left(\frac{5\pi}{24}+\frac{\pi}{24}\right) - \sin\left(\frac{5\pi}{24}-\frac{\pi}{24}\right)\right] = \frac{1}{2}\left[\sin\left(\frac{6\pi}{24}\right) - \sin\left(\frac{4\pi}{24}\right)\right] = \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \frac{1}{2}\right] = \frac{\sqrt{2}-1}{4}$$

6B.) (4 pts) Simplify: $\sin(2\theta)\sin(8\theta)$ using a product-to-sum formula. Write with positive angles.

6B. $\frac{1}{2}[\cos(6\theta) - \cos(10\theta)]$

First rewrite as $\sin(8\theta)\sin(2\theta)$. Use $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$$= \frac{1}{2}[\cos(8\theta - 2\theta) - \cos(8\theta + 2\theta)] = \frac{1}{2}[\cos(6\theta) - \cos(10\theta)]$$

7A.) (3 pts) Find the exact value of $\cos 15^\circ - \cos 75^\circ$ using a sum-to-product formula.

7A. $\frac{\sqrt{2}}{2}$

First switch the order by factoring out a negative: $-(\cos 75^\circ - \cos 15^\circ)$

Then use $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.

$$= -1 \cdot -2\sin\left(\frac{75+15}{2}\right)\sin\left(\frac{75-15}{2}\right)$$

$$= 2\sin(45^\circ)\sin(30^\circ)$$

$$= 2\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{2}$$

7B.) (3 pts) Simplify: $\sin 2x - \sin 7x$ using a sum-to-product formula. 7B. $-2\sin\left(\frac{5x}{2}\right)\cos\left(\frac{9x}{2}\right)$

Write with positive angles.

First switch the order by factoring out a negative: $-(\sin 7x - \sin 2x)$

Then use $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$.

$$= -1\cdot 2\sin\left(\frac{7x-2x}{2}\right)\cos\left(\frac{7x+2x}{2}\right)$$

$$= -2\sin\left(\frac{5x}{2}\right)\cos\left(\frac{9x}{2}\right)$$

8A.) (5 pts) Solve for x: $\sqrt{3}\sec^2 x - 2\sec x = 0$ on $[0,\ 360^\circ)$ 8A. $x = 30^\circ,\ 330^\circ$

$$\sec x\left(\sqrt{3}\sec x - 2\right) = 0$$

$\sec x = 0$ and $\sqrt{3}\sec x - 2 = 0$

$\frac{1}{\cos x} = 0$ $\sec x = \frac{2}{\sqrt{3}}$

$\cancel{\cos x = \frac{1}{0}}$ $\cos x = \frac{\sqrt{3}}{2}$

$x = 30^\circ,\ 330^\circ$

8B.) (5 pts) Solve for θ: $\tan\theta\sin\theta + \sin\theta = 0$ on $[0,\ 2\pi)$ 8B. $\theta = 0,\ \frac{3\pi}{4},\ \pi,\ \frac{7\pi}{4}$

$$\sin\theta(\tan\theta + 1) = 0$$

$\sin\theta = 0$ and $\tan\theta + 1 = 0$

$\theta = 0,\ \pi$ $\tan\theta = -1$

$\theta = \frac{3\pi}{4},\ \frac{7\pi}{4}$

9A.) (5 pts) Solve for x: $2\cos^2 x - \sin x - 1 = 0$ on $[0,\ 2\pi)$ 9A. $x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{3\pi}{2}$

$2\cos^2 x - \sin x - 1 = 0$ Use the identity $\cos^2 x = 1 - \sin^2 x$.

$2(1 - \sin^2 x) - \sin x - 1 = 0$

$2 - 2\sin^2 x - \sin x - 1 = 0$

$-2\sin^2 x - \sin x + 1 = 0$ Multiply both sides by -1

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x - 1 = 0$ and $\sin x + 1 = 0$

$\sin x = \frac{1}{2}$ and $\sin x = -1$

$x = \frac{\pi}{6},\ \frac{5\pi}{6}$ and $x = \frac{3\pi}{2}$

9B.) (5 pts) Solve for θ: $\cos\theta \sin 2\theta = \sin\theta$ on $[0,\ 360^\circ)$ 9B. $\theta = 0^\circ,\ 45^\circ,\ 135^\circ,\ 180^\circ,\ 225^\circ,\ 315^\circ$

$\cos\theta \sin 2\theta = \sin\theta$ Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

$\cos\theta \cdot 2\sin\theta\cos\theta = \sin\theta$

$2\sin\theta\cos^2\theta - \sin\theta = 0$

$\sin\theta(2\cos^2\theta - 1) = 0$

$\sin\theta = 0$ and $2\cos^2\theta - 1 = 0$

$\theta = 0^\circ,\ 180^\circ$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\sqrt{\frac{1}{2}}$

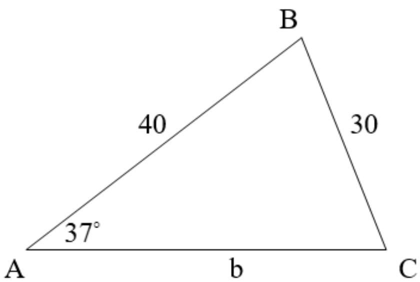
$\cos\theta = \pm\frac{\sqrt{2}}{2}$, $\theta = 45^\circ,\ 135^\circ,\ 225^\circ,\ 315^\circ$

After testing each solution, our answer is $\theta = 0^\circ,\ 45^\circ,\ 135^\circ,\ 180^\circ,\ 225^\circ,\ 315^\circ$.

10A.) (6 pts) Given: a = 30, c = 40, and $m\angle A = 37^\circ$. # of solutions: 2

How many solutions does this triangle have?

Find the following (if possible). Round to two decimal places.



$$\frac{\sin 37^\circ}{30} = \frac{\sin C}{40}$$

$$30\sin C = 40\sin 37^\circ$$

$$\sin C = \frac{40\sin 37^\circ}{30}$$

$$\sin C = 0.8024...$$

$$C_1 \approx 53.36$$

$$C_2 \approx 180 - 53.36 \approx 126.64$$

Since $126.64^\circ + 37^\circ < 180^\circ$, there are two solutions.

$m\angle B_1 = 89.64^\circ$

$m\angle C_1 = 53.36^\circ$

$b_1 \approx 49.9$

$m\angle B_2 = 16.36^\circ$

$m\angle C_2 = 126.64^\circ$

$b_2 \approx 14.04$

Triangle 1:

$$m\angle B_1 = 180^\circ - 37^\circ - 53.36^\circ = 89.64^\circ$$

$$\frac{\sin 37^\circ}{30} = \frac{\sin 89.64^\circ}{b_1}$$

$$b_1 = \frac{30\sin 89.64^\circ}{\sin 37^\circ} \approx 49.85$$

Triangle 2:

$$m\angle B_2 = 180^\circ - 37^\circ - 126.64^\circ = 16.36^\circ$$

$$\frac{\sin 37^\circ}{30} = \frac{\sin 16.36^\circ}{b_2}$$

$$b_2 = \frac{30\sin 16.36^\circ}{\sin 37^\circ} \approx 14.04$$

10B.) (6 pts) Given: a = 81, b = 62, and $m\angle A = 43^\circ$.

How many solutions does this triangle have?

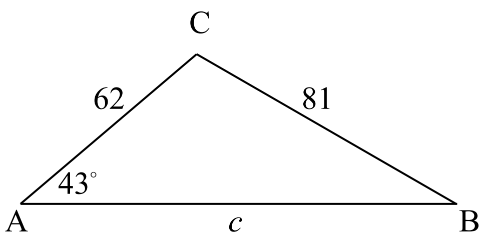
Find the following (if possible). Round to two decimal places.

\# of solutions: 1

$m\angle B_1 = 31.47^\circ$

$m\angle C_1 = 105.53^\circ$

$c_1 \approx 114.43$



$$\frac{\sin 43^\circ}{81} = \frac{\sin B}{62}$$

$$81\sin B = 62\sin 43^\circ$$

$$\sin B = \frac{62\sin 43^\circ}{81}$$

$$\sin B = 0.52202...$$

$$B_1 \approx 31.47$$

$$B_2 \approx 180 - 31.47 \approx 148.53$$

Since $148.52^\circ + 43^\circ > 180^\circ$, there is only one solution.

Triangle 1:

$$m\angle C_1 = 180^\circ - 43^\circ - 31.47^\circ = 105.53^\circ$$

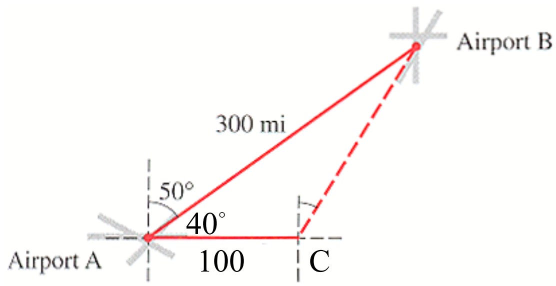
$$\frac{\sin 43^\circ}{81} = \frac{\sin 105.53^\circ}{c_1}$$

$$c_1 = \frac{81\sin 105.53^\circ}{\sin 43^\circ} \approx 114.43$$

11A.) (6 pts) Airport B is 300 miles from airport A at a bearing of $N50^\circ E$ (see figure). A pilot wishing to fly from A to B mistakenly flies 100 miles due east, when he notices the error. How far is the pilot from his destination at the time he notices the error? Round to two decimal places.

11A. 232.46 *miles*

We will first find angle A inside the triangle, which is $90^\circ - 50^\circ = 40^\circ$. Then AC is labeled with 100. Next we use the Law of Cosines.

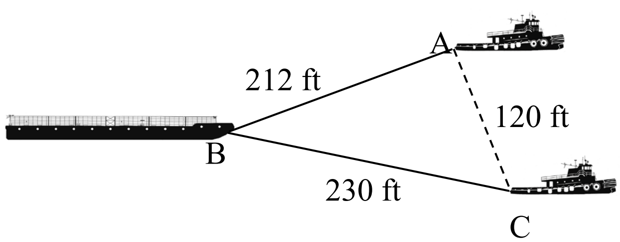


$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 100^2 + 300^2 - 2(100)(300)\cos 40^\circ$$
$$a^2 = 54037.333$$
$$a = 232.46 \text{ miles}$$

11B.) (6 pts) Two tugboats are 120 ft. apart pull a barge, as shown below. If the length of one cable is 212 ft and the length of the other is 230 ft, find the angle formed by the two cables. Round to two decimal places.

11B. 31.17°

First label each ship with a letter:



Now use the Law of Cosines. In our picture we want to find angle B.

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$120^2 = 230^2 + 212^2 - 2(230)(212)\cos B$$
$$-83444 = -97520\cos B$$
$$0.85566... = \cos B$$
$$31.17^\circ = B$$