NAME:____________KEY________________

MATH 181 TEST 3 SAMPLE

NOTE: The actual exam will only have 10 questions. The different parts of each question (part A, B, etc.) are variations. Know how to do all the variations on this exam.

1A.) (6 pts) Find the derivative: $y = \sin^{-1}\left(\sqrt{2}\cdot x\right) - \sec^{-1}\left(\frac{x}{2}\right)$

1A. $y' = \frac{\sqrt{2}}{\sqrt{1-2x^2}} - \frac{2}{|x|\sqrt{x^2-4}}$

$$y' = \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}x\right)^2}} - \frac{\frac{1}{2}}{\left|\frac{1}{2}x\right|\sqrt{\left(\frac{1}{2}x\right)^2-1}} = \frac{\sqrt{2}}{\sqrt{1-2x^2}} - \frac{\frac{1}{2}}{\frac{1}{2}|x|\sqrt{\frac{1}{4}x^2-1}} = \frac{\sqrt{2}}{\sqrt{1-2x^2}} - \frac{1}{|x|\sqrt{\frac{x^2-4}{4}}}$$

$$= \frac{\sqrt{2}}{\sqrt{1-2x^2}} - \frac{1}{\frac{|x|\sqrt{x^2-4}}{2}} = \frac{\sqrt{2}}{\sqrt{1-2x^2}} - \frac{2}{|x|\sqrt{x^2-4}}$$

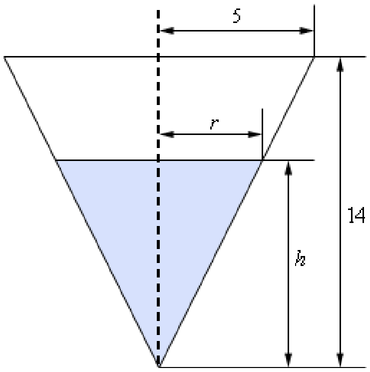
1B.) (6 pts) Find the derivative: $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1}(x)$

1B. 0

$$y' = -\frac{\frac{-1}{x^2}}{1+\left(\frac{1}{x}\right)^2} - \frac{1}{1+x^2} = \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} - \frac{1}{1+x^2} = \frac{\frac{1}{x^2}}{\frac{x^2+1}{x^2}} - \frac{1}{1+x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

2A.) (6 pts) A tank of water in the shape of a cone is leaking water at a constant rate of $2\ ft^3/hour$. The base radius of the tank is 5 ft and the height of the tank is 14 ft. At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft? Use $V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$. Hint: similar triangles.

2A. $\frac{dh}{dt} = -\frac{98}{225\pi}\ ft/s$



Given: $\frac{dV}{dt} = -2\ ft^3/hr$, $h = 6$, $\frac{dh}{dt} = ?$

Similar Triangles:

$$\frac{r}{h} = \frac{5}{14} \qquad V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$$

$$14r = 5h \qquad V = \frac{1}{3} \cdot \pi \cdot \left(\frac{5}{14}h\right)^2 \cdot h$$

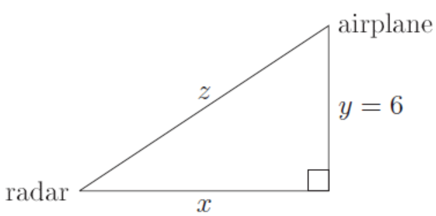
$$r = \frac{5}{14}h \qquad V = \frac{25\pi}{588} \cdot h^3$$

$$V = \frac{25\pi}{588} \cdot h^3$$

$$\frac{dV}{dt} = \frac{25\pi}{196} \cdot h^2 \frac{dh}{dt} \Rightarrow -2 = \frac{25\pi}{196} \cdot 6^2 \frac{dh}{dt} \Rightarrow -2 = \frac{225\pi}{49} \frac{dh}{dt} \Rightarrow -\frac{98}{225\pi} = \frac{dh}{dt}$$

2B.) (6 pts) An airplane is flying over a radar tracking system station at a height of 6 miles. Suppose the distance is decreasing at a rate of 400 miles per hour. What is the velocity of the plane when the distance is 10 miles?

2B. $-500\ mph$



$$x^2 + 6^2 = z^2$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

Given: $\frac{dz}{dt} = -400\ mph$

$$2(8)\frac{dx}{dt} = 2(10)(-400)$$

$\frac{dx}{dt} = ?$

$$\frac{dx}{dt} = -500\ mph$$

$z = 10\ miles \Rightarrow x = 8\ miles$

3A.) (6 pts) The height of a triangle is increasing at a rate of 3 inches per minute while the base of the triangle is decreasing at a rate of 2 inches per minute. At the instant when the height is 8 inches and the base is 4 inches, what is the rate of change of the area of the triangle? NOTE: $A = \frac{1}{2}bh$

3A. $\frac{dA}{dt} = -2 \ in^2/\text{min}$

Given: $\frac{dh}{dt} = 3$, $\frac{db}{dt} = -2$, $h = 8$, $b = 4$

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}b \cdot \frac{dh}{dt} + h \cdot \frac{1}{2}\frac{db}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2}(4)(3) + \frac{1}{2}(8)(-2)$$

$$\frac{dA}{dt} = -2 \ in^2/\text{min}$$

3B.) (6 points) The mechanics at Lincoln Automotive are reboring a 5 inch deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius 0.0002 inches per minute while not changing the depth of the cylinder. How rapidly is the cylinder's volume changing when the radius is 1.7 inches? Note: $V = \pi \cdot r^2 \cdot h$.

3B. $0.0034\pi \ in^3/\text{min}$

Given: $\frac{dr}{dt} = 0.0002$, $r = 1.7$, $\frac{dh}{dt} = 0$, $h = 5$

$$\frac{dV}{dt} = \pi \cdot r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r\frac{dr}{dt}$$

$$\frac{dV}{dt} = \pi \cdot 1.7^2 \cdot 0 + 5 \cdot 2\pi(1.7)(0.0002)$$

$$\frac{dV}{dt} = 0.0034\pi \ in^3/\text{min}$$

4A.) (6 pts) Let $f(\theta) = \sin\theta + \cos\theta$ on $[0,\ 2\pi]$. Find all critical numbers. Then find the absolute extrema on this interval.

Critical numbers: $\frac{\pi}{4}, \frac{5\pi}{4}$

Max: $\sqrt{2}$ Occurs at: $x = \frac{\pi}{4}$

Min: $-\sqrt{2}$ Occurs at: $x = \frac{5\pi}{4}$

$f'(\theta) = \cos\theta - \sin\theta$

$0 = \cos\theta - \sin\theta$

$\cos\theta = \sin\theta$

Critical Numbers: $\frac{\pi}{4}, \frac{5\pi}{4}$

x	0	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	2π
f(x)	1	$\sqrt{2}$	$-\sqrt{2}$	1

4B.) (6 pts) Let $f(x) = x^2 - 8\ln x$ on [1, 4]. Find all critical numbers. Then find the absolute extrema on this interval.

Critical number(s): $x = 2$

Max: $16 - 8\ln 4$ Occurs at: 4

Min: $4 - 8\ln 2$ Occurs at: 2

$f'(x) = 2x - \frac{8}{x}$

$0 = 2x - \frac{8}{x}$

$2x = \frac{8}{x}, \Rightarrow 2x^2 = 8, \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Critical Number: 2

-2 is not in the domain, so it is not a critical number. The derivative is undefined at 0, however 0 is not in the domain, so it is not a critical number.

x	1	2	4
f(x)	1	$4 - 8\ln 2$	$16 - 8\ln 4$

5A.) (6 pts) Find all values of c that satisfy the equation $\frac{f(b)-f(a)}{b-a}=f'(c)$ in the conclusion of the Mean Value Theorem if $f(x)=\sqrt{x(2-x)}$ on [0, 2].

5A. $c=1$

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$$\frac{f(2)-f(0)}{2-0}=\frac{1}{2}\left(2c-c^2\right)^{-\frac{1}{2}}(2-2c)$$

$$\frac{0-0}{2}=\frac{2-2c}{2\sqrt{c(2-c)}}$$

$0=4-4c$, so $c=1$.

5B.) (6 pts) Find all values of c that satisfy the equation $\frac{f(b)-f(a)}{b-a}=f'(c)$ in the conclusion of the Mean Value Theorem if $f(x)=x^2+2x-1$ on [0, 1].

5B. $c=\frac{1}{2}$

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$$\frac{f(1)-f(0)}{1-0}=2c+2$$

$$\frac{2-(-1)}{1}=2c+2$$

$3=2c+2$, so $c=\frac{1}{2}$

6A.) (8 pts) Use $y=\frac{4x}{x^2+9}$ to determine the interval(s) of increasing / decreasing and the relative (local) extrema.

Increasing: (–3, 3)

Decreasing: $(-\infty,-3)\cup(3,\infty)$

Relative Max: $\left(3,\ \frac{2}{3}\right)$

Relative Min: $\left(-3,\ -\frac{2}{3}\right)$

$$y'=\frac{(x^2+9)(4)-(4x)(2x)}{\left(x^2+9\right)^2}=\frac{4x^2+36-8x^2}{\left(x^2+9\right)^2}=\frac{36-4x^2}{\left(x^2+9\right)^2}$$

$$0=\frac{36-4x^2}{\left(x^2+9\right)^2}\Rightarrow 0=36-4x^2\Rightarrow 4x^2=36\Rightarrow x^2=9\Rightarrow x=\pm 3$$

–	+	–

–3 3

6B.) (8 pts) Use $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ to determine the interval(s) of increasing / decreasing and the relative (local) extrema.

Increasing: $(-1,\ 0)\cup(0,\ \infty)$

Decreasing: $(-\infty,\ -1)$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} + \frac{4}{3}x^{-\frac{2}{3}} \quad \Rightarrow \quad f'(x) = \frac{4x^{\frac{1}{3}}}{3} + \frac{4}{3x^{\frac{2}{3}}}$$

Relative Max: None

0 makes $f'(x)$ undefined, but 0 is defined in $f(x)$.

Relative Min: (–1, –3)

$$0 = \frac{4x^{\frac{1}{3}}}{3} + \frac{4}{3x^{\frac{2}{3}}} \quad \Rightarrow \quad \frac{-4x^{\frac{1}{3}}}{3} = \frac{4}{3x^{\frac{2}{3}}} \quad \Rightarrow \quad -12x = 12 \quad \Rightarrow \quad x = -1$$

–	+	+

–1 0

6C.) (8 pts) Use $f(x) = \frac{(\sin x + 1)^2}{2}$ to determine the interval(s) of increasing / decreasing and the relative (local) extrema on $[0,\ 2\pi)$.

Increasing: $\left(0,\ \frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},\ 2\pi\right)$

Decreasing: $\left(\frac{\pi}{2},\ \frac{3\pi}{2}\right)$

$$f'(x) = (\sin x + 1)\cdot \cos x$$

Relative Max: $\left(\frac{\pi}{2}, 2\right)$

$$0 = (\sin x + 1)\cdot \cos x \quad \Rightarrow \quad \cos x = 0 \quad \text{and} \quad \sin x + 1 = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Relative Min: $\left(\frac{3\pi}{2}, 0\right)$

+	–	+

0 $\frac{\pi}{2}$ $\frac{3\pi}{2}$ 2π

7A.) (6 pts) Use $f(x)=x^3(3x^2+20x+40)$ to find the interval(s) of concavity and inflection pts.

Concave up: $(0,\infty)$

Concave down: $(-\infty,0)$

Inflection point(s): $(0,0)$

$$f(x)=3x^5+20x^4+40x^3$$
$$f'(x)=15x^4+80x^3+120x^2$$
$$f''(x)=60x^3+240x^2+240x$$
$$0=60x^3+240x^2+240x$$
$$0=60x(x^2+4x+4)$$
$$0=60x(x+2)(x+2),\ \text{so}\ x=0,\ -2$$

–	–	+
	–2	0

7B.) (6 pts) Use $f(x)=\tan x+2x$ on $\left(\frac{\pi}{2},\ \frac{3\pi}{2}\right)$ to find the interval(s) of concavity and inflection pts.

Concave up: $\left(\pi,\ \frac{3\pi}{2}\right)$

Concave down: $\left(\frac{\pi}{2},\ \pi\right)$

Inflection point(s): $(\pi,\ 2\pi)$

$$f(x)=\tan x+2x$$
$$f'(x)=\sec^2 x+2$$
$$f''(x)=2\sec x\cdot\sec x\tan x$$
$$0=2\sec^2 x\tan x$$
$$0=2\sec^2 x\quad\text{or}\quad 0=\tan x$$
$$\cancel{0=\frac{2}{\cos^2 x}}\quad\text{or}\quad x=\pi$$

–	+
$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

8A.) (6 pts) Sketch the curve $f(x)$ that meets the following conditions:

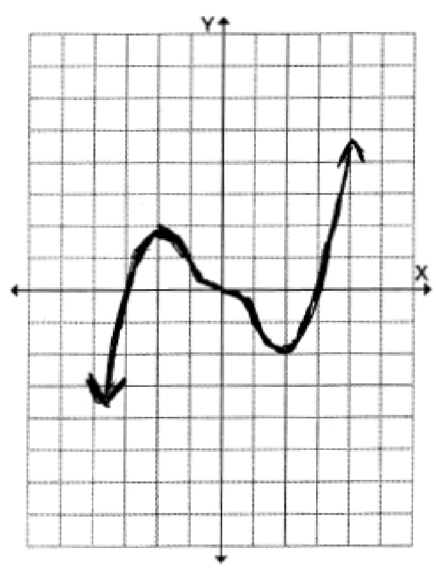
$$f(-3)=f(0)=f(3)=0$$
$$f'(-2)=f'(0)=f'(2)=0$$
$$f''(-1)=f''(0)=f''(1)=0$$

Sign changes for $f'(x)$

+	–	–	+
	–2	0	2

Sign changes for $f''(x)$

–	+	–	+
	–1	0	1



8B.) (6 pts) Sketch the curve $f(x)$ that meets the following conditions:

$f(-4) = f(0) = f(4) = 0$
$f'(-3) = f'(0) = f'(3) = 0$
$f''(-2) = f''(2) = 0$

Sign changes for $f'(x)$

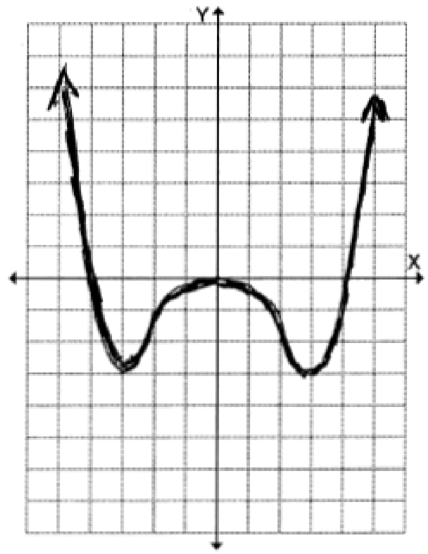
–	+	–	+

–3 0 3

Sign changes for $f''(x)$

+	–	+

–2 2



9A.) (4 pts) Use L'Hopital's rule to evaluate the limit: $\lim_{x\to 0}\left[\frac{xe^{4x}-x}{1-\cos(2x)}\right]$ 9A. 2

$$\lim_{x\to 0}\left[\frac{xe^{4x}-x}{1-\cos(2x)}\right] = \lim_{x\to 0}\left[\frac{xe^{4x}(4)+e^{4x}(1)-1}{1-(-\sin(2x))(2)}\right] = \lim_{x\to 0}\left[\frac{4xe^{4x}+e^{4x}-1}{1+2\sin(2x)}\right] = \lim_{x\to 0}\left[\frac{4xe^{4x}(4)+e^{4x}(4)+e^{4x}(4)}{2\cos(2x)(2)}\right]$$

$$= \lim_{x\to 0}\left[\frac{16xe^{4x}+8e^{4x}}{4\cos(2x)}\right] = \frac{16(0)e^{4(0)}+8e^{4(0)}}{4\cos(2(0))} = \frac{8(1)}{4(1)} = 2$$

9B.) (4 pts) Use L'Hopital's rule to evaluate the limit: $\lim_{x\to 0}\left[\frac{x\sin x}{\cos(3x)-1}\right]$ 9B. $-\frac{2}{9}$

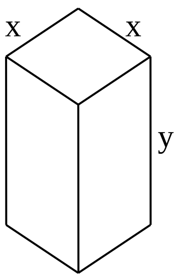
$$\lim_{x\to 0}\left[\frac{x\sin x}{\cos(3x)-1}\right] = \lim_{x\to 0}\left[\frac{x\cos x+\sin x(1)}{-\sin(3x)(3)}\right] = \lim_{x\to 0}\left[\frac{x\cos x+\sin x}{-3\sin(3x)}\right] = \lim_{x\to 0}\left[\frac{x(-\sin x)+\cos x(1)+\cos x}{-3\cos(3x)(3)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-x\sin x+2\cos x}{-9\cos(3x)}\right] = \frac{-(0)\sin(0)+2\cos(0)}{-9\cos(3(0))} = \frac{2(1)}{-9(1)} = -\frac{2}{9}$$

10A.) (6 pts) Your iron works has contracted to design and built a 500 cubic foot, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible (use the least material). What dimensions do you tell the shop to use?

10A. Base: 10 ft by 10 ft

Height: 5 ft



$$V = x^2 y \quad \Rightarrow \quad 500 = x^2 y \quad \Rightarrow \quad \frac{500}{x^2} = y$$

$$S = x^2 + 4xy$$

$$S = x^2 + 4x\left(\frac{500}{x^2}\right)$$

$$S = x^2 + 2000x^{-1}$$

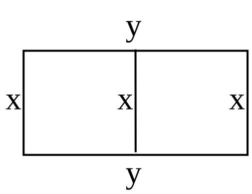
$$S' = 2x - 2000x^{-2}$$

$$0 = 2x - \frac{2000}{x^2} \quad \Rightarrow \quad 2x = \frac{2000}{x^2} \quad \Rightarrow \quad 2x^3 = 2000 \quad \Rightarrow \quad x^3 = 1000 \quad \Rightarrow \quad x = 10 \quad \Rightarrow \quad y = \frac{500}{10^2} = 5$$

10B.) (6 pts) A 216 square meter rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides (see figure). What dimensions for the outer rectangle will require the smallest total length of fence?

10B. 12 m by 18 m

First label the drawing:



$$xy = 216 \quad \Rightarrow \quad y = \frac{216}{x}$$

$$P = 3x + 2y$$

$$P = 3x + 2\left(\frac{216}{x}\right)$$

$$P = 3x + 432x^{-1}$$

$$P' = 3 - 432x^{-2}$$

$$0 = 3 - \frac{432}{x^2} \quad \Rightarrow \quad \frac{432}{x^2} = 3 \quad \Rightarrow \quad 3x^2 = 432 \quad \Rightarrow \quad x^2 = 144 \quad \Rightarrow \quad x = 12 \quad \Rightarrow \quad y = \frac{216}{12} = 18$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\cos^{-1} u\right] = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Derivative of a Natural Logarithm

Let u be a differentiable function of *x*. Then:

1.) $\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$ where $x > 0$

2.) $\frac{d}{dx}\left[\ln u\right] = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$ where $u > 0$

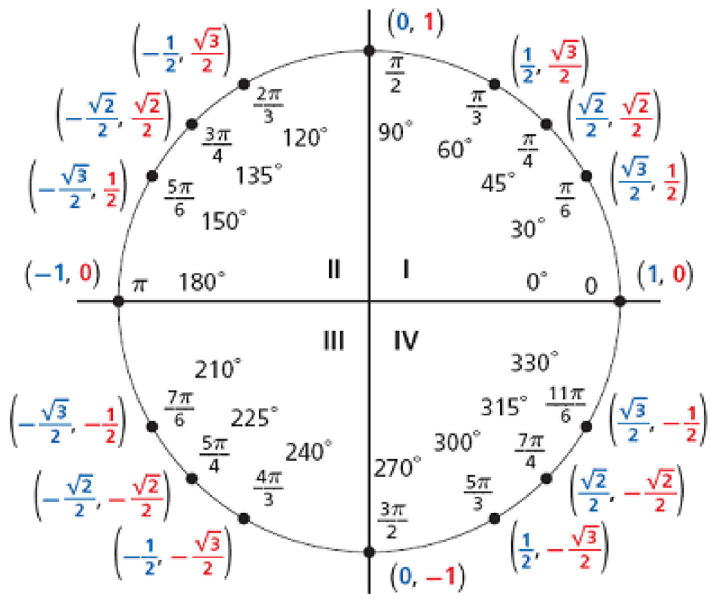
Derivative of a^x

$$\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \cdot u'$$

Derivative of $\log_a x$

$$\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{u \ln a}$$

The Unit Circle



MATH 181 TEST 3 REVIEW PROBS

Section	**Problems**
3.9	#21 – 39
3.10	#11 – 14, 19 – 23, 26, 27, 30, 31, 41
4.1	#21 – 34, 37 – 40 (no graphs)
4.2	#1 – 7 (determine if MVT can be applied and find c)
4.3	#19 – 44 (all parts), 45 – 62 (part a only)
4.4	#9 – 43 (no graphs), 49 – 58 (no graphs), 104, 106
4.5	NONE (just do homework in MML)
4.6	#1, 2, 4 – 9, 11, 13, 14, 15, 16, 18, 20, 23