

11.1 Systems of Linear Equations: Substitution and Elimination

In this section we will solve systems of linear equations, which can be solved using substitution and elimination methods. These are basically equations of lines. You will have three cases with the answers. The first and most common is that the lines will cross at a point and you will have a solution. The second situation is if the lines are parallel. Parallel lines never cross, so there will be no solution. The third situation is if one line is on top of another line. In this case all the points on one line are the same as all the points on the other line, so there are infinite solutions.

Substitution Method

In this method you will take one equation and solve for either x or y . Then you will substitute this into the other equation.

EXAMPLE: Solve the system using the substitution method:

$$\begin{cases} 3x - y = 11 \\ -2x + 5y = -16 \end{cases}$$

You can solve for any variable in any equation to begin this problem. You want to choose the variable that will be the easiest to solve for. I will solve for y in the first equation since there is no number in front of it. If there was a number in front of the y I would need to divide and that would cause fractions. So I will take the equation $3x - y = 11$ and solve for y . You will get $y = 3x - 11$. Now we need to substitute this into the other equation for y :

$$\begin{aligned} -2x + 5(3x - 11) &= -16 && \text{Now solve for } x. \\ -2x + 15x - 55 &= -16 \\ 13x - 55 &= -16 \\ 13x &= 39 \\ x &= 3 \end{aligned}$$

Now that we have x we need to now solve for y . If we go back to our equation for y we have $y = 3x - 11$. We can put a 3 in for x to get y : $y = 3(3) - 11$. So $y = -2$. So for this problem our answer is $x = 3$ and $y = -2$. This means these two lines will cross at the point $(3, -2)$.

EXAMPLE: Solve the system using the substitution method:

$$\begin{cases} 3x + 2y = -9 \\ 2x - 4y = -6 \end{cases}$$

Once again you want to solve for either x or y , whatever is the easiest. I will solve for x in the second equation. If you take $2x - 4y = -6$ and solve for x you will get: $x = 2y - 3$. Now we can put this into the first equation in place of x :

$$\begin{aligned} 3(2y - 3) + 2y &= -9 && \text{Now solve this for } y. \\ 6y - 9 + 2y &= -9 \\ 8y - 9 &= -9 \\ 8y &= 0 \\ y &= 0 \end{aligned}$$

Now that we have y we need to now solve for x . If we go back to our equation for y we have $x = 2y - 3$. We can put a 0 in for x to get x : $x = 2(0) - 3$. So $x = -3$. So for this problem our answer is $x = -3$ and $y = 0$. This means these two lines will cross at the point $(-3, 0)$.

For both of those previous examples we were able to solve for x or y without fractions. Unfortunately this will not always happen. Sometimes you might get a fraction when you solve for x or y . This is the limitation of the substitution method. The elimination method will allow us to solve these type of problems easier.

Elimination Method

In this method we will multiply one or both of the equations by something so that when we add the equations together one of the variables will be eliminated.

EXAMPLE: Solve the system using the elimination method:

$$\begin{array}{r} 2x + 3y = -4 \\ 5x - 6y = -37 \end{array}$$

In this problem we can choose to eliminate the x or y . Suppose we wanted to eliminate the y terms. I want to multiply one or both equations by a number so that when I add the equations together the y terms will cancel. I notice that if I multiply the top equation by 2 I will get a $6y$. When I add $6y$ and $-6y$ they will cancel. To find out what to multiply the equations by you need to find the smallest number that both 3 and 6 will divide into. In this case it is 6, so I want to get one equation to have a 6 and the other a -6 .

Suppose you wanted to cancel the x terms. You want to find the smallest number that both 2 and 5 divide into. This number would be 10. I want one x term to be 10 and the other to be -10 . If I wanted to eliminate the x terms I would multiply the first equation by -5 and the second equation by 2. I will choose to eliminate the y terms since I only need to multiply the top equation by 2:

$$\begin{array}{r} 4x + 6y = -8 \\ 5x - 6y = -37 \end{array}$$

Here I multiplied the whole first row by 2. Now I want to add these equations together.

$$\begin{array}{r} 9x = -45 \\ x = -5 \end{array}$$

When I added the equations together, the y terms were eliminated. Now solve for x .

To solve for y , we can put -5 in for x in either the first or second equation. I will put this into the first equation:

$$\begin{array}{r} 2(-5) + 3y = -4 \\ -10 + 3y = -4 \\ 3y = 6 \\ y = 2 \end{array}$$

Now solve for y .

So our answer is $x = -5$ and $y = 2$. The point of intersection would be $(-5, 2)$.

EXAMPLE: Solve the system using the elimination method:

$$\begin{array}{r} 3x + 4y = -11 \\ 5x + 3y = -11 \end{array}$$

You can either eliminate the x or the y . This time I will eliminate the x terms. There is a 3 and a 5. The smallest number that both 3 and 5 will divide into will be 15. In order for them to cancel I must multiply one of the equations by a negative. It doesn't matter which one you multiply by a negative. I will multiply the top equation by -5 and I will multiply the bottom equation by 3:

$$-15x - 20y = 55$$

$$15x + 9y = -33$$

Now add the equations together:

$$-11y = 22$$

$$y = -2$$

Solve for y.

Now we need to put -2 in for y in either the top or bottom equation. I will use the bottom equation this time:

$$5x + 3(-2) = -11 \quad \text{Solve for x.}$$

$$5x - 6 = -11$$

$$5x = -5$$

$$x = -1$$

So our answer is $x = -1$ and $y = -2$. The point of intersection would be $(-1, -2)$.

EXAMPLE: Solve the system using any method:

$$\begin{aligned} \frac{2}{3}x - y &= \frac{20}{3} \\ 2x - \frac{5}{2}y &= 17 \end{aligned}$$

Before we use either method, let's first get rid of the fractions. We can multiply the first equation by 3 since this is the common denominator. We can also multiply the second equation by 2 to get rid of the equations.

$$2x - 3y = 20$$

$$4x - 5y = 34$$

The elimination method would be easier to solve. Multiply the top equation by -2 :

$$-4x + 6y = -40$$

$$4x - 5y = 34$$

Now add the equations together.

$$y = -6$$

When you put this back to get x , you can use the non-fraction equations we found. I will use $2x - 3y = 20$ and put in -6 for y : $2x - 3(-6) = 20$. Solving this we get $x = 1$. So our answer is $x = 1$ and $y = -6$, or $(1, -6)$.

EXAMPLE: Solve the system using any method:

$$\begin{aligned} 3x - y &= 4 \\ 6x - 2y &= 7 \end{aligned}$$

This could be solved by either substitution or elimination. I will solve it by substitution. We can solve the first equation for y . You will get $y = 3x - 4$. We want to substitute this into the bottom equation.

$$6x - 2(3x - 4) = 7$$

$$6x - 6x + 8 = 7$$

$$8 = 7$$

We didn't do anything wrong here. We ended up with a false statement, which means this system has no solution. These lines are parallel, so they don't cross.

EXAMPLE: A party mix is made by adding nuts that sell for \$2.50 per kg to a cereal mixture that sells for \$1 per kg. How much of each should be added to get 30 kg of a mix that will sell for \$1.70 per kg.

For this problem we need to find our two equations and then use either substitution or elimination to solve it. The best way to do these kind of problems is to set up a table:

	Number of Kilograms	Price per Kilogram	Value
Nuts			
Cereal			
Mixture			

We will let x = the number of kg of nuts. Then y = the number of kg of cereal. We have 30 kg of the mixture. We know the price per kg of nuts is \$2.50. The price per kg of cereal is \$1.00. The price per kg for the mixture is \$1.70. Let's put all of this information on the table. For the value column I just multiplied the number of kg by the price per kg.

	Number of Kilograms	Price per Kilogram	Value
Nuts	x	2.50	$2.5x$
Cereal	y	1.00	y
Mixture	30	1.70	$30(1.70) = 51$

We will set up two equations. One will be for the number of kg and the other will be for the value. I know that Nuts + Cereal = Mixture, so we have:

$$x + y = 30$$

$$2.5x + y = 51$$

Now we have our system that we can solve by either substitution or elimination. I will solve this by elimination. I can multiply the first row by -1:

$$-x - y = -30$$

$$2.5x + y = 51$$

Now add the equations together.

$$1.5x = 21$$

$$x = 14 \text{ kg.}$$

Solve for x .

We can put this into the first equation to find y : $14 + y = 30$. Solving this we get $y = 16$ kg. Therefore we must mix 14 kg of nuts and 16 kg of cereal.

EXAMPLE: A chemist working on a flu vaccine needs to mix a 10% sodium-iodine solution with a 60% sodium-iodine solution to obtain 50 ml of a 30% sodium-iodine solution. How many ml of each should be mixed?

Once again a table would work best here. We will let $x = 10\%$ solution, $y = 60\%$ solution. We are given that there must be 50ml of the 30% mixture. We will put this information on the table. To get the amount of sodium-iodine we must multiply the number of ml by the percent written as a decimal.

	Number of ml	Percent (as a decimal)	Amount of Sodium-iodine
10% solution	x	0.1	$0.1x$
60% solution	y	0.6	$0.6y$
30% solution mixture	50	0.3	$50(0.3) = 15$

One of our equations will be from the number of ml. The other equation will come from the amount of sodium-iodine:

$$x + y = 50$$

$$0.1x + 0.6y = 15$$

I will solve this one by substitution. Take the first equation and solve for y : $y = 50 - x$. Then put this into the second equation and solve for x :

$$0.1x + 0.6(50 - x) = 15 \quad \text{Solve for } x.$$

$$0.1x + 30 - 0.6x = 15$$

$$-0.5x + 30 = 15$$

$$-0.5x = -15$$

$$x = 30 \text{ ml}$$

Since $y = 50 - x$, then $y = 50 - 30$, so $y = 20\text{ml}$.

The chemist must mix 30ml of the 10% solution with 20ml of the 60% solution.