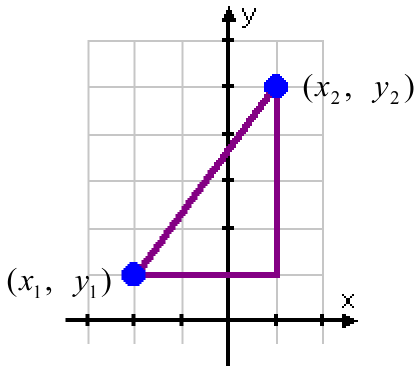
1.3 Lines

Slope Formula

The slope formula is used to find the slope between two points (x_1, y_1) and (x_2, y_2).



The slope is the vertical change divided by the horizontal change. From our picture, the vertical change is $y_2 - y_1$ and the horizontal change is $x_2 - x_1$.

From this we get the formula for slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

- Positive slopes will increase as you move from left to right.
- Negative slopes will decrease as you move from left to right.
- A slope of zero is a horizontal line.
- An undefined or infinity slope is a vertical line.

EXAMPLE: Find the slope of a line passing through the following points. Indicate whether the line increases, decreases, is horizontal or vertical.

a.) (-1, 3) and (2, 4)

To do this problem we can label our point so we know what to put into the slope formula. It doesn't matter which point you call x_1 or x_2. I will label the point as the following: $x_1 = -1$, $y_1 = 3$, $x_2 = 2$, $y_2 = 4$. Now we plug these into the slope formula: $m = \dfrac{4-3}{2-(-1)} = \dfrac{1}{3}$. Since the slope is positive we know this line increases.

b.) (4, -1) and (3, -1)

First we label the point as the following: $x_1 = 4$, $y_1 = -1$, $x_2 = 3$, $y_2 = -1$. Now we plug these into the slope formula: $m = \dfrac{-1-(-1)}{3-4} = \dfrac{0}{-1} = 0$. Since the slope is zero we know this line is horizontal.

c.) (3, -2) and (3, -5)

First we label the point as the following: $x_1 = 3$, $y_1 = -2$, $x_2 = 3$, $y_2 = -5$. Now we plug these into the slope formula: $m = \dfrac{-5-(-2)}{3-3} = \dfrac{-3}{0} = \textit{undefined}$. Since the slope is undefined we know this line is vertical.

Slope-Intercept Formula– this is the standard form of a line which allows you to easily identify the slope and y-intercept.

$y = mx + b$ Here the slope is m and the y-intercept is (0, b).

Linear Function– this is the same as the slope-intercept form, except with function notation. In general, a linear function begins with $f(x)$ and contains an x with a power of 0 or 1.

$f(x) = mx + b$

Point-Slope Formula – this is used when you want to find the equation of a line when you are given a slope and another point on the line. This other point does not need to be the y-intercept.

$y - y_1 = m(x - x_1)$

EXAMPLE: Use the information and given conditions to write an equation for each line in slope-intercept form as well as the point-slope form.

a.) Slope = 8, passing through (4, -1).

For this one, we know that m = 8, $x_1 = 4$, and $y_1 = -1$. We can plug these into our point-slope formula: $y - (-1) = 8(x - 4)$. When we simplify, we get: $y + 1 = 8(x - 4)$. The equation of this line is now written in point-slope form, which is one of our answers. Now we need to write it in slope-intercept form. To do this, we just need to solve for y. First we distribute the 8: $y + 1 = 8x - 32$. Now subtract 1 from both sides to get our second answer: $y = 8x - 33$.

b.) Slope = $-\frac{3}{5}$, passing through (10, -4).

For this one, we know that m = $-\frac{3}{5}$, $x_1 = 10$, and $y_1 = -4$. We can plug these into our point-slope formula: $y - (-4) = -\frac{3}{5}(x - 10)$. When we simplify, we get: $y + 4 = -\frac{3}{5}(x - 10)$. The equation of this line is now written in point-slope form, which is one of our answers. Now we need to write it in slope-intercept form. First we distribute the $-\frac{3}{5}$: $y + 4 = -\frac{3}{5}x - \left(\frac{3}{5}\right)\left(-\frac{10}{1}\right)$. To multiply the two fractions on the end, multiply across the top and bottom. You will get: $y + 4 = -\frac{3}{5}x + 6$. Now subtract 4 from both sides to get our second answer: $y = -\frac{3}{5}x + 2$.

c.) Passing through (-3, 6) and (3, -2)

This time we are not given a slope, so we first must use the slope formula. We label our points and put them into the slope formula: $m=\frac{-2-6}{3-(-3)}=\frac{-8}{6}=-\frac{4}{3}$. When we use the point-slope formula we can use EITHER of our given points as the $(x_1,\ y_1)$. In this case I will use the first point. So $x_1=-3$, and $y_1=6$. We can plug these into our point-slope formula:
$y-6=-\frac{4}{3}(x-(-3))$. When we simplify we get: $y-6=-\frac{4}{3}(x+3)$. The equation of this line is now written in point-slope form, which is one of our answers. Now we need to write it in slope-intercept form. First, we distribute the $-\frac{4}{3}$: $y-6=-\frac{4}{3}x-4$. Now add 6 to both sides to get our second answer: $y=-\frac{4}{3}x+2$.

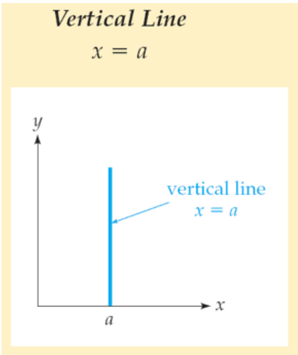
d.) x-intercept = $-\frac{1}{2}$, y-intercept = 4

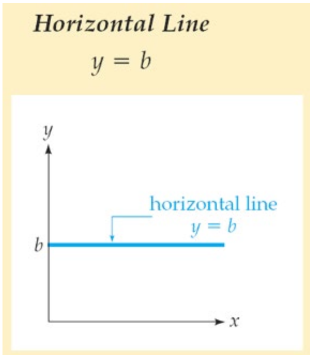
This time we are not given a slope, so again we must use the slope formula. We want to put these intercepts in a point form, which would be: $\left(-\frac{1}{2},\ 0\right)$ and $(0,\ 4)$. We label our points and put them into the slope formula:
$m=\frac{4-0}{0-\left(-\frac{1}{2}\right)}=\frac{4}{\frac{1}{2}}=8$. When we use the slope-intercept we can use EITHER of our two given points for x and y. The reason for using the second point is that since this is the y-intercept, we already know that $b = 4$. Since we also have our m, we are ready to write this in slope-intercept form: $y=8x+4$.

Horizontal and Vertical lines

$x=a$ is a vertical line crossing the x-axis at $x=a$.

$y=b$ is a horizontal line crossing the y-axis at $y=b$.





EXAMPLE: Write the equation of a vertical line containing the point (–2,3).

A vertical line has the form $x = a$. The a represents the x-value of the point the line passes through. Therefore, the equation of this line is $x = –2$.

EXAMPLE: Write the equation of a horizontal line containing the point (5,4).

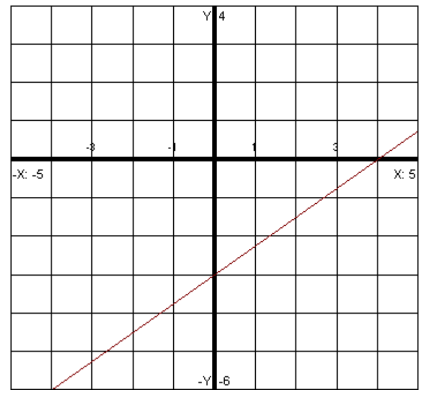
A horizontal line has the form $y = b$. The b represents the y-value of the point the line passes through. Therefore, the equation of this line is $y = 4$.

EXAMPLE: Write the following in slope-intercept form and identify the slope and y-intercept. Use this information to graph the equation.

a.) $f(x) = \frac{3}{4}x - 3$

This equation can be written in slope-intercept form by replacing $f(x)$ with y. You will get: $y = \frac{3}{4}x - 3$.

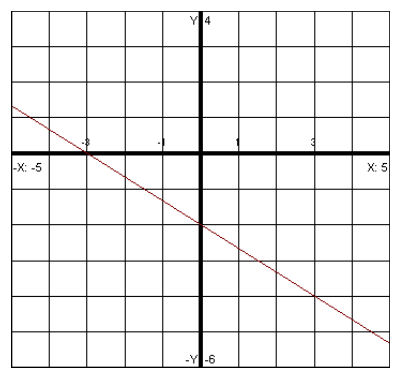
From here we can identify that the slope is $\frac{3}{4}$ and the y-intercept is (0, -3).



To graph this, first plot the y-intercept. Now we need another point on our line. The definition of the slope is the change in the vertical distance divided by the change in the horizontal distance. In our slope the top number is 3. This is our vertical change. Because it is positive, we will move up 3 units from our y-intercept. The bottom number is 4, so we will need to move 4 units to our right. So from our y-int we will move up 3 units and 4 units to the right. This will give us our next point. Plot this and connect our two points with a line.

b.) $4x + 6y = -12$

We need to solve for y in order to put this into slope-intercept form. First isolate y: $6y = -4x - 12$. Now divide both sides by 6 to get: $y = -\frac{2}{3}x - 2$. Now we can identify that the slope is $-\frac{2}{3}$ and the y-intercept is (0, -2). To graph this, first plot the y-intercept. Now the fraction $-\frac{2}{3}$ can be written as either $\frac{-2}{3}$ or $\frac{2}{-3}$. If we think of the slope as $\frac{-2}{3}$ then the our vertical change is -2. This means we move DOWN 2 units from our y-intercept. The bottom number is 3, so we will need to move 3 units to our right. So from our y-int we will move DOWN 2 units and 3 units to the right. This will give us our next point. Plot this and connect our two points with a line. If we used $\frac{2}{-3}$ then we would move UP two units and to the LEFT 3 units. Notice we will still get another point on the same line, so we can use either fraction.



Parallel lines have the same slope. These lines do not cross.

Perpendicular lines have opposite reciprocal slopes (opposite sign and one fraction is flipped over). Ex: $\frac{4}{3}$ and $-\frac{3}{4}$ are opposite reciprocals. Also $-\frac{1}{2}$ and 2 are opposite reciprocals. Perpendicular lines will cross at a right angle (90 degrees). Another characteristic is if you multiply together the two slopes of perpendicular lines, you will always get -1.

EXAMPLE: Find the slope of a line that is perpendicular to $y = -4x - 3$.

The slope of this line is –4, and because it says perpendicular, we need to find the opposite reciprocal. The number -4 can be rewritten as the fraction $-\frac{4}{1}$. Because it is a negative, the opposite sign will be positive. If we flip over the fraction we get $\frac{1}{4}$, which is our answer.

EXAMPLE: Determine whether the following lines are parallel, perpendicular, or neither.

$$3x - 4y = 12$$
$$8y - 6x = 16$$

For the first equation, we will solve for y: $-4y = -3x + 12 \quad \Rightarrow \quad y = \frac{3}{4}x - 3$

For the second equation, we will solve for y: $8y = 6x + 16 \quad \Rightarrow \quad y = \frac{6}{8}x + \frac{16}{8} \quad \Rightarrow \quad y = \frac{3}{4}x + 2$

These slopes are the same, so they are parallel.

EXAMPLE: Use the given conditions to write an equation for each line in point-slope form and slope-intercept form.

a.) Passing through (-2, -7) and parallel to the line whose equation is $y = -5x + 4$

Since we want a line parallel, then this will have the same slope as our given equation, so we know m = -5. We are also given a point. From here, we will plug this information into the point-slope formula. You should get: $y - (-7) = -5(x - (-2))$. Simplifying gives you $y + 7 = -5(x + 2)$. This is our first answer. To get this into the slope-intercept form, we need to solve for y. First distribute: $y + 7 = -5x - 10$. Now subtract 7 from both sides: $y = -5x - 17$. This is our answer in slope-intercept form.

b.) Passing through (-4, 2) and perpendicular to the line whose equation is $y = \frac{1}{3}x + 7$

Since we want a line perpendicular, then this will have the opposite reciprocal slope as our given equation, so we know m = -3. We are also given a point. From here, we will plug this information into the point-slope formula. You should get: $y - 2 = -3(x - (-4))$. Simplifying gives you $y - 2 = -3(x + 4)$. This is our first answer. To get this into the slope-intercept form, we need to solve for y. First distribute: $y - 2 = -3x - 12$. Now add 2 to both sides: $y = -3x - 10$. This is our answer in slope-intercept form.

c.) Passing through (5, -9) and perpendicular to the line whose equation is $x+7y-12=0$

Since we want a line perpendicular, then this will have the opposite reciprocal slope as our given equation. We first need to solve our given equation for y: $y=-\frac{1}{7}x+\frac{12}{7}$. The slope of this line is $-\frac{1}{7}$. A line perpendicular to this one will have a slope of m = 7. From here, we will our slope and given point into the point-slope formula. You should get: $y-(-9)=7(x-5)$. Simplifying gives you: $y+9=7(x-5)$. This is our first answer. To get this into the slope-intercept form, we need to solve for y. First distribute: $y+9=7x-35$. Now subtract 9 from both sides: $y=7x-44$. This is our answer in slope-intercept form.

EXAMPLE: The cost of renting a truck is \$20 a day plus 50 cents per mile. Write a linear equation that relates the cost C, in dollars, of renting the truck to the number of x miles driven. What is the total cost if you drive 160 miles?

First we need the equation for the cost. No matter how many miles are driven you must pay \$20 for the one day rental. Added to this is 50 cents per mile. The cost equation is C = 0.5x + 20 where x is the number of miles driven. Next, we will plug in 160 miles for x. So, $C=0.5(160)+20=\$100$. Therefore, the total cost of renting the truck after driving for 160 miles is \$100.

EXAMPLE: A manufacturer buys a new machine costing \$120,000. It is estimated that the machine has a useful lifetime of 10 years, and a salvage value of \$4000 at that time. Find a linear function for the value, V, of the machine after t years.

Since it is linear, we know we will have a line. In order to get the equation of the line we need a slope and a point. We don't have a slope here, but we can find it by setting up two points and then using the slope formula. The time is going to represent x since and value is the y. This is because the value depends on the time, just like y depends on x. If the machine is new, then t = 0. At that time the value is \$120,000 since it is new. The point is (0, 120000). We need another point. We are told that after 10 years the machine is worth \$4000. Our point is (10, 4000). Now we have our two points so we can use the slope formula

$m=\frac{120000-4000}{0-10}=\frac{116000}{-10}=-11600$. This means that the machine's value drops by \$11600 each year.

Now to find the equation we will use y = mx + b. We know m but we need x and y. Just use either point that we started with. I will use (10, 4000):

$y=mx+b$
$4000=(-11600)(10)+b$
$b=120000$

If we put these together into the equation we will get: $V(t)=-11600t+120000$ which is the answer.

EXAMPLE: A company is planning to manufacture a certain product. The fixed costs will be \$500000 and it will cost \$400 to produce each product. Each will be sold for \$600. What is the profit equation and how many units must be sold in order to break even?

Profit is defined as the revenue minus the costs. We need to find our revenue and cost equations. The costs involve a fixed price plus a variable price. The equation is $C = 400x + 500000$. Since each is sold for \$600 then this is the revenue, which is price times quantity. You will get $R = 600x$. To get the profit function you need to subtract the cost from the revenue: $P = 600x - (400x + 500000)$. Simplifying you get: $P = 200x - 500000$. When you break even the profit will be zero. Put a zero in for P and solve for x: $0 = 200x - 500000$. Solving this you will get $x = 2500$ units.