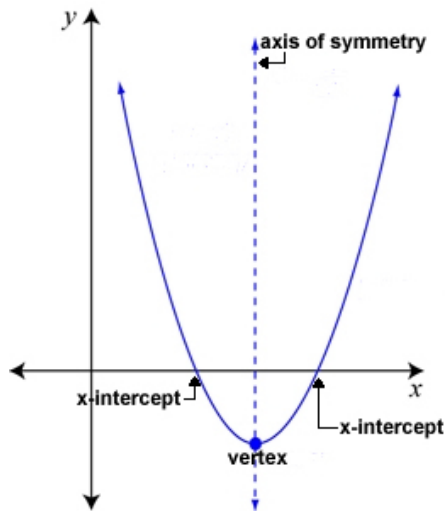


3.3 Quadratic Functions

This section is all about quadratic functions, which give U shaped graphs called parabolas. First we need to define a couple of terms involving parabolas:



At the bottom of this graph we have the vertex. This is either the lowest or highest point on a parabola. In the standard form $y = ax^2 + bx + c$ if $a > 0$ then the graph opens up and the vertex is the lowest point on the graph. If $a < 0$ then the graph opens down and the vertex is the highest point on the graph.

Another new term is the axis of symmetry. This is a vertical line that always goes through the x-coordinate of the vertex. Since this is a vertical line it will start with $x =$. This is considered a fold line since the parabola can be folded on top of itself.

Vertex form: $y = a(x - h)^2 + k$. If an equation is written in this form, then the vertex is (h, k) . If the a value is negative, then the graph opens down. If the a value is positive, then the graph opens up.

EXAMPLE: Find the vertex, axis of symmetry, intercepts and graph of $y = -2(x + 3)^2 + 8$.

In order to find h , we can rewrite the formula as: $y = -2(x - (-3))^2 + 8$.

Now we know $h = -3$. We also know that $k = 8$. So the vertex is $(-3, 8)$.

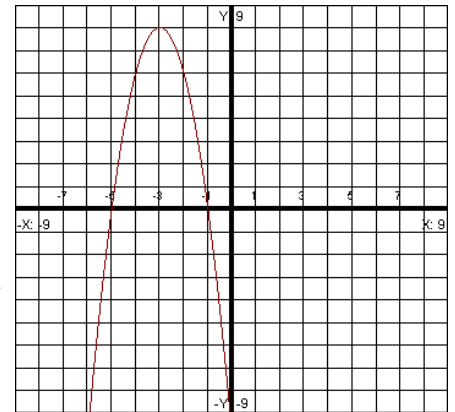
The axis of symmetry goes through the x-coordinate of the vertex, so this equation is $x = -3$. To find the y-intercept, put in a zero for x in our original equation: $y = -2(0 + 3)^2 + 8$. Solving this, we get $y = -10$. The y-intercept is written as $(0, -10)$. For the x-intercept, put in a zero for y :

$0 = -2(x + 3)^2 + 8$. To solve this, subtract 8 from both sides: $-8 = -2(x + 3)^2$.

Now divide both sides by -2 : $4 = (x + 3)^2$. Now take the square root of both sides to get: $\pm 2 = x + 3$. This gives us two separate equations to solve:

$x + 3 = 2$ and $x + 3 = -2$. Solving each individually we get $x = -1, -5$.

We write the x-intercept as $(-1, 0)$ and $(-5, 0)$. Plot all these points to get the graph.



If you have a graph given in vertex form, you can also graph it using transformations like we learned before.

So now what do we do if the quadratic equation is not written in vertex form? The book shows that we can start with the standard form of a quadratic and use the completing the square method to find a formula to find the vertex if an equation is not written in vertex form. I will just give the formula here.

Vertex formula:

Given $y = ax^2 + bx + c$, then the x-coordinate of the vertex is $x = \frac{-b}{2a}$. To find the y-coordinate put the x back into the original equation.

EXAMPLE: Find the vertex by using the formula: $y = 3x^2 + 6x + 1$. Then find the axis of symmetry.

Here, $a = 3$, $b = 6$, and $c = 1$.

First we use the formula: $x = \frac{-b}{2a} = \frac{-6}{2(3)} = -1$. So now we know that $x = -1$. Now we need to put -1 in for x in the

original equation y : $y = 3(-1)^2 + 6(-1) + 1$. After simplifying we get $y = -2$. We write the vertex as $(-1, -2)$. This will be an actual point on the parabola. The a in this problem is greater than zero, therefore the vertex will be a minimum.

The axis of symmetry is equal to the x -coordinate of the vertex. To find it, just put $x =$ and then the x -coordinate of the vertex. For this problem the axis of symmetry is $x = -1$.

EXAMPLE: Find the vertex by using the formula: $y = -2x^2 - 5x + 1$. Then find the axis of symmetry.

Here, $a = -2$, $b = -5$ and $c = 1$. Using the vertex formula we get: $x = \frac{-(-5)}{2(-2)} = -\frac{5}{4}$. We get a fraction. This

doesn't matter. We still put this in for x in the y equation: $y = -2\left(\frac{-5}{4}\right)^2 - 5\left(\frac{-5}{4}\right) + 1$. To square a fraction

you need to square the top and bottom number you will get: $y = -2\left(\frac{25}{16}\right) - 5\left(\frac{-5}{4}\right) + 1$. Now multiply across

the top and across the bottom when multiplying fractions: $y = \frac{-25}{8} + \frac{25}{4} + 1$. We need common denominators.

$y = \frac{-25}{8} + \frac{50}{8} + \frac{8}{8}$ Now we can add the fractions to get: $y = \frac{33}{8}$. So the vertex is $\left(-\frac{5}{4}, \frac{33}{8}\right)$. Since a is less than zero, the vertex will be a maximum. The axis of symmetry is $x = -\frac{5}{4}$.

EXAMPLE: Determine, without graphing, whether the quadratic function $y = -2x^2 + 8x + 3$ has a maximum value or a minimum value and then find the value.

Since $a < 0$, we know we will have a maximum value. Now we need to find it using the vertex formula.

$x = -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$. Now plug this back into the equation to find the y : $y = -2(2)^2 + 8(2) + 3$.

Simplifying gives $y = -8 + 16 + 3$, so we know $y = 11$. Therefore the maximum value is 11.

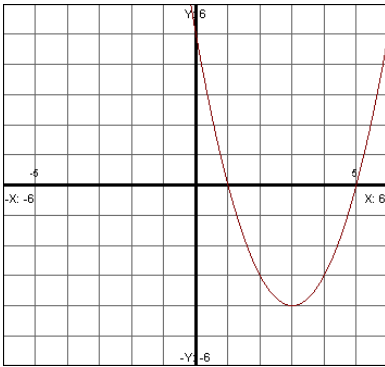
Now we will do problems that involve graphing the parabola. You will need to find the x and y intercepts and the vertex first. Then you can plot the points and get the graph.

EXAMPLE: Find the intercepts, vertex, axis of symmetry, domain, range, and the graph of $y = x^2 - 6x + 5$.

First we will find the x-intercept. Put in a zero for y. You will get $0 = x^2 - 6x + 5$. In order to solve this you must factor. You will get $0 = (x-1)(x-5)$. Solving you will get $x = 1$ and $x = 5$, or $(1, 0)$ and $(5, 0)$ in intercept form. To find the y-intercept, put in a 0 for x. You will get $y = 5$, or $(0, 5)$ in intercept form. Now we need to find the vertex. We will use the vertex formula so find the x coordinate of the vertex:

$$x = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3. \text{ So now we will put in a 3 for x in the original equation to get the y-value of the vertex.}$$

$$y = (3)^2 - 6(3) + 5 \text{ so } y = -4. \text{ The vertex is } (3, -4). \text{ Now we just need to plot the points to get the graph.}$$



Notice that the number in front of the x^2 is greater than zero, so the parabola opens up and the vertex is the lowest point on the graph.

Here the domain (x values) is: $(-\infty, \infty)$.

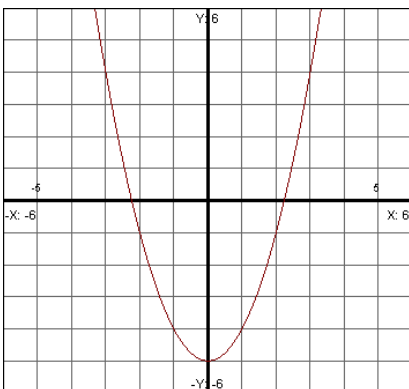
The range (y values) is $[-4, \infty)$

EXAMPLE: Find the intercepts, vertex, axis of symmetry, domain, range, and the graph of $y = x^2 - 5$.

When we solve $0 = x^2 - 5$ to find the x-intercept we can use the square root method. Add 5 to both sides. You will get $x^2 = 5$. Take the square root of both sides to get $x = \pm\sqrt{5}$, or $(\pm\sqrt{5}, 0)$ in intercept form. To find the y-intercept, put in a 0 for x. You will get $y = -5$, or $(0, -5)$ in intercept form. Now we need to find the vertex.

We will use the vertex formula so find the x coordinate of the vertex. Here the b is zero: $x = \frac{-b}{2a} = \frac{0}{2(1)} = 0$.

So now we will put in a 0 for x in the original equation to get the y-value of the vertex: $y = 0^2 - 5 = -5$ so the vertex is $(0, -5)$. To graph we will plot the points we found. To plot $(\pm\sqrt{5}, 0)$ we can change this into a decimal so we know where this is on our x-axis. The decimal form is: $(\pm 2.24, 0)$.



The number in front of the x^2 is greater than zero, so the parabola opens up and the vertex is a minimum.

Here the domain (x values) is: $(-\infty, \infty)$.

The range (y values) is $[-5, \infty)$.

EXAMPLE: Find the intercepts, vertex, axis of symmetry, domain, range, and the graph of $y = 6 - 4x + x^2$.

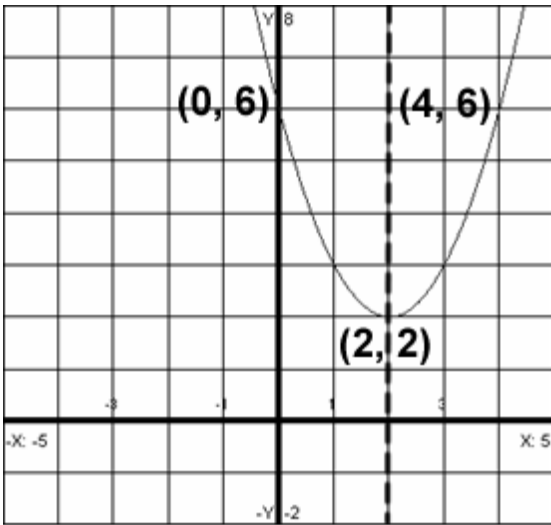
Putting in a zero for x will give us the y -intercept, which is $(0,6)$. To find the x -intercept, put in a 0 for y . You will get: $0 = x^2 - 4x + 6$. We need to use the quadratic formula since this one does not factor. Notice I rewrote the problem with descending powers. Now we know $a = 1$, $b = -4$, and $c = 6$. If you put this into the quadratic

formula, you will get: $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(6)}}{2(1)} = \frac{4 \pm \sqrt{16 - 24}}{2} = \frac{4 \pm \sqrt{-8}}{2} = \frac{4 \pm 2i\sqrt{2}}{2} = 2 \pm i\sqrt{2}$. Notice

this is not a real number. This means the graph does not have any x -intercepts, so we know it does not cross the x -axis. Now we need to find the vertex. We will use the vertex formula so find the x coordinate of the vertex.

$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2$. So now we will put in a 2 for x in the original equation to get the y -value of the vertex:

$y = 6 - 4(2) + (2)^2 = 6 - 8 + 4 = 2$ so the vertex is $(2, 2)$. Since we have the vertex then we can also find the axis of symmetry, which is $x = 2$. To graph we will plot the points we found, which is $(0, 6)$ and $(2, 2)$. Now we need to find a third point. This is where we can use the axis of symmetry. We can draw a vertical line through the graph once we plot our two points. Let's look at the point $(0, 6)$. This is a distance of 2 units away from the axis of symmetry. We know that there is another point two units to the right of the axis of symmetry that has the same y -value as $(0, 6)$. We will get the point $(4, 6)$. Notice that this graph is completely above the x -axis, meaning it does not have any REAL x -intercepts, however it does have two complex intercepts: $2 \pm i\sqrt{2}$



The domain is $(-\infty, \infty)$.

The range is $[2, \infty)$.

Modeling with Quadratic Functions

These problems involve finding maximum or minimum values. ALL of the problems in this section require you to find the vertex once you have your equation.

To find a maximum or minimum value you must find the vertex.

EXAMPLE: Suppose the height of an object shot straight up is given by $h = 512t - 16t^2$ where h is measured in feet and t is in seconds. Find the maximum height and the time at which the object hits the ground.

This problem is easier than the previous ones since we don't need to first figure out the formula. You want to find the vertex. This will give us the time at which the maximum height will occur. Use the vertex formula:

$$t = \frac{-512}{2(-16)} = 16. \text{ So we know it takes 16 seconds for the object to reach its maximum height. To find the}$$

height, put in a 16 for each t in $h = 512t - 16t^2$. You will get: $h = 512(16) - 16(16)^2 = 4096 \text{ ft}$.

The second part of the problem asks us to find the time at which the object hits the ground. This occurs when the height is zero. So put in a zero for h and solve for t : $0 = 512t - 16t^2$. We will solve by factoring.

$$0 = 512t - 16t^2$$

$0 = 16t(32 - t)$ Solving this you will get $t = 32$ seconds. You also get $t = 0$ as a solution.

EXAMPLE: Among all pairs of numbers whose difference is 24, find a pair whose product is as small as possible. What is the minimum product?

We need to set up two equations on this one. The word DIFFERENCE means subtraction, so one equation is $x - y = 24$. The word product means multiply, so another formula is $P = xy$. We need to solve the first equation for either x or y and then substitute this into the second equation. If we add y to both sides in the first equation we will get $x = y + 24$. Now put this into the second equation in place of x . We will get $P = (y + 24)y$. Distributing will give us $p = y^2 + 24y$. Now we use the vertex formula. It doesn't matter that we have y

instead of x . Just means we will find the y first: $y = \frac{-24}{2(1)} = -12$. So we know that $y = -12$. To find the

second number, we will use the formula $x = y + 24$. This means $x = -12 + 24 = 12$. So our pair of numbers is 12 and -12. If we multiply these together we will get the product: $12(-12) = -144$.

EXAMPLE: The daily revenue, R , achieved by selling x boxes of candy is figured to be $R(x) = 9.5x - 0.04x^2$. The daily cost, C , of selling x boxes of candy is $C(x) = 1.25x + 250$. How many boxes must be sold to maximize the profit? What will be the maximum profit?

First we need to find the profit function. Remember that Profit = Revenue - Costs.

$$P(x) = R(x) - C(x)$$

$$P(x) = 9.5x - 0.04x^2 - (1.25x + 250)$$

$$P(x) = 9.5x - 0.04x^2 - 1.25x - 250$$

Remember to distribute the negative. Now simplify.

$$P(x) = -0.04x^2 + 8.25x - 250$$

We have our function, so now apply the vertex formula.

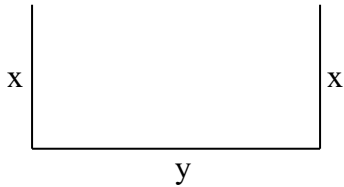
$$x = \frac{-8.25}{2(-0.04)} = 103.125 \quad \text{We need to round up to 104 boxes. This is how many must be sold in order to}$$

maximize the profit. Now we need to find out what the maximum profit will be. This is done by putting 104 into for x in our profit equation: $P(x) = -0.04(104)^2 + 8.25(104) - 250$. We work this out to get our answer:

$$P(x) = -0.04(10816) + 858 - 250 = -432.64 + 858 - 250 = \$175.36. \text{ So our maximum profit will be } \$175.36 \text{ when we sell 104 boxes.}$$

EXAMPLE: Suppose you have 900 ft of fencing and you want to enclose a rectangular field that borders a river, so only three sides require fencing. What is the largest area that can be enclosed?

Start by drawing a picture and labeling the sides. We only have three sides to this rectangle:



First we want to find a formula for the perimeter. It is $900 = 2x + y$. Solving for y we get $y = 900 - 2x$. Now substitute $900 - 2x$ for y in the area formula $A = xy$. You will get $A(x) = x(900 - 2x)$, or $A(x) = 900x - 2x^2$. Since it is asking you for the largest area, we need to find the vertex. We will use the vertex formula.

$x = \frac{-900}{2(-2)} = 225$. So we know that $x = 225$. To find the y value we will use the formula $y = 900 - 2x$ and put in 225 for x . You will get $y = 900 - 2(225) = 450$. So $y = 450$. The question is asking us to find the largest area. We will multiply x and y to get the answer. Our area will be 225 multiplied by 450, which is 101250 ft^2 .