4.1 Polynomial Functions

Polynomial Function: $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_o$

The n in the formula above is called the **degree**, and this is the largest exponent of the polynomial. A polynomial can only have whole number exponents (no negatives, fractions or decimals). A polynomial must also be a smooth line with no breaks or corners.

The a_n is always in the term with the degree. In other words, it is the number in front of the x with the highest power. More on this later…

EXAMPLE: Indicate whether the following are polynomials. If they are, indicate the degree and the a_n.

a.) $f(x) = 3x - 2x^3 + \frac{x^2}{3}$

This is a polynomial since the exponents are all whole numbers. The degree is the highest power that you see, which in this case is 3. The a_n is -2 since it is the number that comes in front of the x^3 term.

b.) $f(x) = \sqrt{x} - 5$

This is not a polynomial because a root is a fractional power.

c.) $f(x) = \frac{5}{x^2}$

This is not a polynomial since this can be written as $f(x) = 5x^{-2}$. Polynomials can never have a negative exponent.

d.) $f(x) = 6$

This is a polynomial. This can also be written as $f(x) = 6x^0$. So the degree is zero since that is the highest power. The a_n is 6.

e.) $f(x) = (x-2)(x+5)$

We can multiply this out to get $f(x) = x^2 + 3x - 10$. This is a polynomial of degree 2. The a_n is 1.

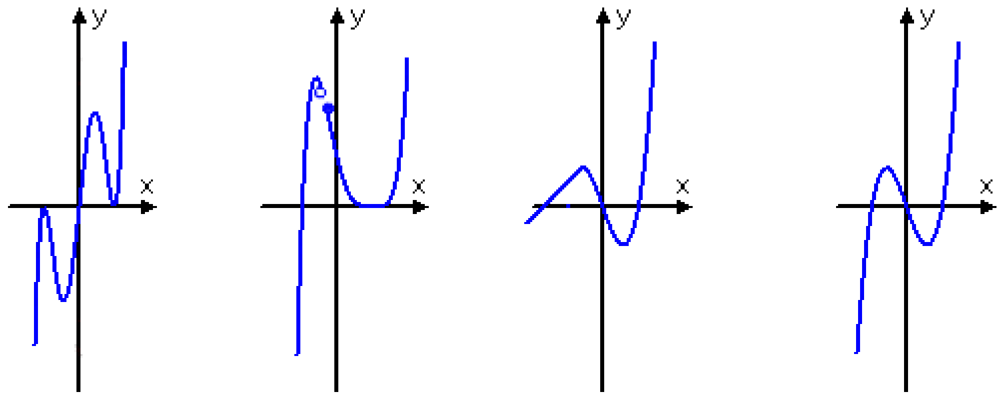
Turning point – a point in which the graph changes directions. This happens at a peak or valley.

If n is the degree of a polynomial then the polynomial can have <u>at most</u> n – 1 turning points.

EXAMPLE: Up to how many turning points can $y = x^2 - x^5$ have?

Since the degree is 5, then the polynomial can have <u>at most</u> 5 – 1, or 4 turning points. It is important to keep in mind that it will not have exactly 4 turning points. It means that 4 is the most turning points possible.

EXAMPLE: Which of the following can be a degree 3 polynomial?



A B C D

Graph A has four turning points, which is too many for a degree 3 polynomial. A degree 3 polynomial can have AT MOST 3 – 1 turning points. Graph B has a break in the graph, and this can't be a polynomial since they must be smooth continuous curves. Graph C has a corner or "cusp" which is the official math terminology. Polynomials can't have cusps. So choice D is the only one that can be a degree 3 polynomial.

The text looks at the graphs of $y = x^4$ and $y = x^5$. We will skip this portion. You do not need to do any of the homework problems that have you do transformations with these graphs

If r is an x-intercept of a graph y, then x – r is a factor.

EXAMPLE: Form a degree 3 polynomial whose zeros are -2, 0, 2 passing through the point (–1,6).

In order to get the polynomial we must first find its factors. According to the rule above x minus the zero is a factor. So we can form three factors using our zeros: $y = a \cdot (x-(-2))(x-0)(x-2)$. This can be rewritten as: $y = a \cdot x(x+2)(x-2)$. There will be a coefficient, a, but we do not know what this is yet. First, let's multiply this out to verify it is a degree 3 polynomial. You will get:

$y = a \cdot x(x^2 - 4)$

$y = a\left(x^3 - 4x\right)$ Notice that the highest power is 3, so the degree is 3.

To find *a*, plug in our point. We will put in a –1 for *x* and a 6 for *y*: $6 = a\left((-1)^3 - 4(-1)\right)$. Now simplify: $6 = 3a$, so we get $a = 2$. Therefore, our equation is $y = 2\left(x^3 - 4x\right)$ or $y = 2x^3 - 8x$.

EXAMPLE: Form a degree 3 polynomial whose zeros are -3, and 1 with a coefficient of 1.

We can form our two factors to get: $y = (x-(-3))(x-1)$ which is also $y = (x+3)(x-1)$. What is wrong with this answer? The problem is that if we multiply out our answer we only get a degree 2. We can make it a degree 3 by doing either $y = (x+3)^2(x-1)$ or $y = (x+3)(x-1)^2$. This gives us the same zeros but it will also allow our degree to be 3.

The 2 that I added to either of those factors is called the **multiplicity**. *Multiplicity* is basically the power on each factored piece. Usually you will indicate what the zero is and then classify its multiplicity.

If the zero has an even multiplicity, the graph will ***touch*** the x-axis at that zero but it will not pass through. If the zero has an odd multiplicity the graph will ***cross*** the x-axis at that zero.

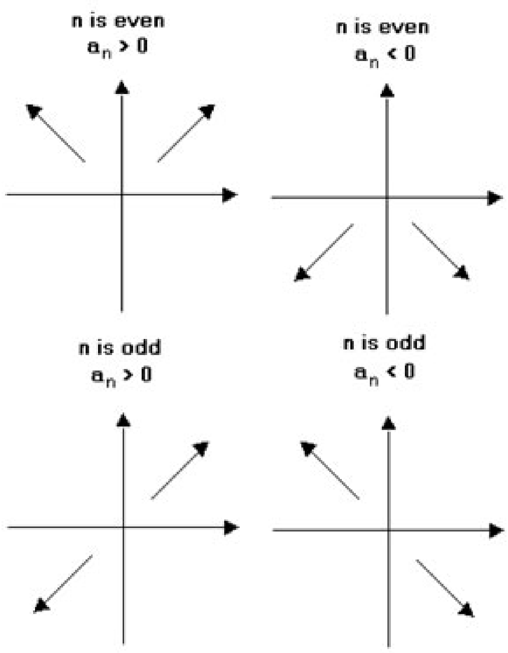
EXAMPLE: Indicate the zeros and multiplicities of each zero from $y = 2x^3(x-1)^2(x+2)^4$. Also indicate whether the graph crosses or touches the x-axis at this zero.

We set the first factor equal to zero we get 0. This has a multiplicity of 3.
We set the second factor equal to zero and we get 1. This has a multiplicity of 2.
We set the third factor equal to zero and we get –2. This has a multiplicity of 4.

The graph will cross at x = 0. The graph will touch at both $x = 1$ and $x = -2$.

End behavior

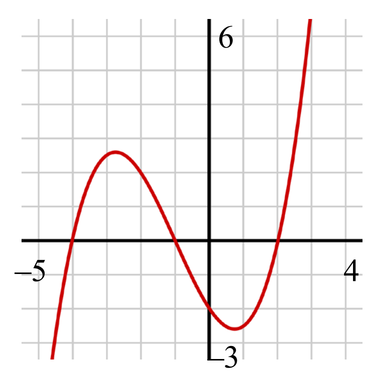
This is what the graph will do when x is really big or really small. This is where you will need to know what the a_n is. Depending on what the degree is and what the a_n is the graph will do the following:



The n is the degree. So this chart is giving us all the possibilities of how the graph will end.

Some textbooks also refer to this end behavior as a "power function of degree n". This is a monomial function of the form $f(x) = ax^n$ where a is a real number, $a \neq 0$, and $n > 0$ is an integer. This is what equation the graph resembles for very large or very small values of x.

EXAMPLE: Construct a polynomial function that has the given graph below.



To do this problem we must first find the zeros. This is where the graph crosses the x-axis. This occurs at –4, –1, and 2. If r is a zero, then $x - r$ is a factor. We will use this to construct the polynomial. Our factors are: $(x-(-4))(x-(-1))(x-2)$. Since the graph crosses at each zero, we know that the multiplicities of each factor is 1. Therefore, our polynomial is: $f(x) = a(x+4)(x+1)(x-2)$ where a is the leading coefficient. To find the a value, we must use a point that is on the graph. Let's use (0, –2). We will plug in a 0 for x and a –2 for $f(x)$. Then solve for a:
$-2 = a(0+4)(0+1)(0-2)$. Simplifying gives us $-2 = a(4)(1)(-2)$ so $-2 = -8a$. Solving gives us a = 1/4, so our polynomial is:
$f(x) = \frac{1}{4}(x+4)(x+1)(x-2)$