4.3 Properties of Rational Functions

Rational Function: a function whose numerator and denominator are polynomials

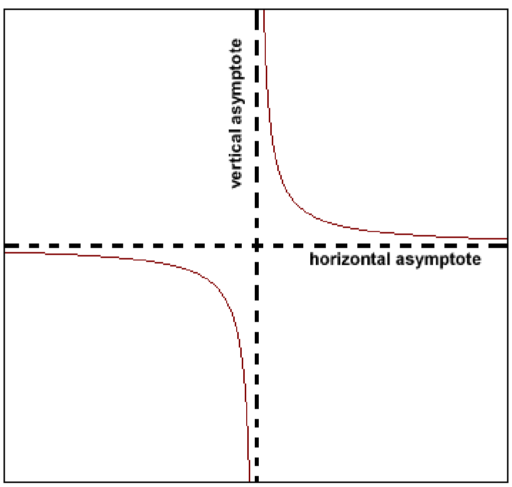
To find the y-intercept for a rational function, put in a zero for x.

To find the x-intercept for a rational function, set the numerator equal to zero

EXAMPLE: Find the x and y intercepts for $y = \frac{2x-12}{x+2}$

First we will find the y-intercept. We will put in a zero for x. You will get: $y = \frac{2(0)-12}{0+2} = \frac{-12}{2} = -6$. So our point is (0, -6). Next we will find the x-intercept. To do this we must set the numerator (top) equal to zero. You will get: $2x-12=0$. Solving this you will get x = 6. So our point is (6, 0).

Asymptote: describes the behavior of a graph as x or y approaches infinity. There are two types of asymptotes. There is a vertical and horizontal asymptote as show in the picture below. The vertical asymptote has an equation that starts with x = since this is a vertical line. The horizontal asymptote has an equation y = since this is a horizontal line. Now we will show how you find these algebraically.



To find the vertical asymptote:
set the denominator equal to zero and solve for x.

Suppose we wanted to find the vertical asymptote of our previous example, $y = \frac{2x-12}{x+2}$. To do this we need to set the denominator equal to zero, so we will have x + 2 = 0. The equation will be x = -2. You would write your answer in this form. You don't need parenthesis. That is only for intercepts.

To find the horizontal asymptote:

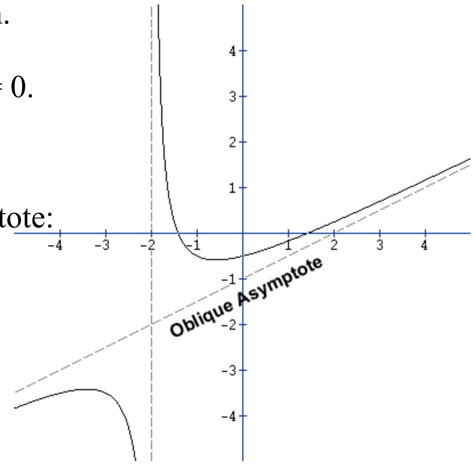
First we need to define some variables. Let's look at the general form of rational expression:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots.}{b_m x^m + b_{m-1}x^{m-1} + \ldots.}$$

Let n be the highest power (degree) of the numerator.
Let m be the highest power (degree) of the denominator.

Let a_n be the number that comes in front of the x with the highest power in the numerator.
Let b_m be the number that comes in front of the s with the highest power in the denominator.

In order to determine the horizontal asymptote we need to look at the n and m.



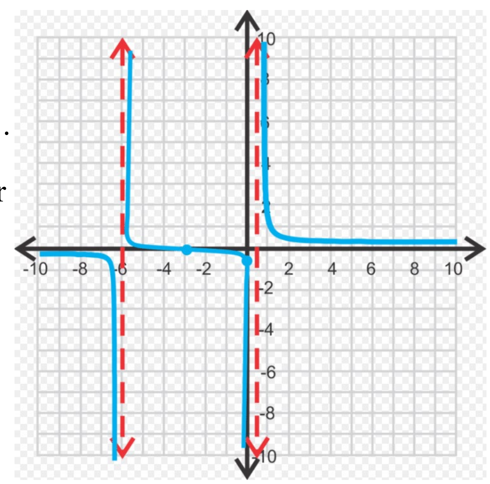
1.) If n < m then the equation of the horizontal asymptote is automatically y = 0.

2.) If n = m then the equation of the horizontal asymptote is $y = \frac{a_n}{b_m}$.

3.) If n > m then there is no horizontal asymptote. There is an oblique asymptote: You find the oblique asymptotes by using long division. More on this later.

In our original equation $y = \frac{2x-12}{x+2}$, if we were asked to find the equation of the horizontal asymptote we need to see which rule applies. Here the highest power on top is the same as the highest power on the bottom, so rule 2 applies. We know $a_n = 2$ and $b_m = 1$, so the horizontal asymptote is $y = \frac{2}{1}$ or just y = 2.

EXAMPLE: Use the graph shown to find the following: (a) The domain and range of the function; (b) The intercepts, if any; (c) horizontal asymptotes, if any; (d) vertical asymptotes, if any; (e) oblique asymptotes, if any.



(a) A vertical asymptote occurs where there is division by zero, and this will not be included in the domain. The domain is all numbers not including –6 and $\frac{1}{2}$. The interval notation is $(-\infty,-6)\cup\left(-6,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$. The range is all the y-values that are used. In this case the graph will never use the y-value of 0 since there is a horizontal asymptote. Therefore the range is all numbers not including 0: $(-\infty,0)\cup(0,\infty)$.

(b) There is an x-intercept at (–3,0), but no y-intercept.

(c) The horizontal asymptote is $y = 0$.

(d) The vertical asymptotes are $x = -6$ and $x = \frac{1}{2}$.

(e) There are no oblique asymptotes.

EXAMPLE: Find the intercepts and asymptotes but DO NOT GRAPH: $y = \frac{1-x^2}{x^2-5x+6}$. State the domain.

First you want to factor to see if it can be simplified further: $y = \frac{(1-x)(1+x)}{(x-2)(x-3)}$. This can't be simplified anymore, so now let's find the y-intercept. To do this we need to put in a 0 for x. When you do that you will get $y = \frac{(1-0)(1+0)}{(0-2)(0-3)} = \frac{1}{6}$. So the y-intercept is $\left(0,\frac{1}{6}\right)$. To find the x-intercept we need to set the top equal to zero. So we have $(1-x)(1+x)=0$. Solving this we get $x = \pm 1$, so our x-intercepts are $(\pm 1,0)$.

Now let's find the asymptotes. To find the vertical asymptote we need to set the bottom equal to zero. So we have $(x-2)(x-3)=0$ so $x = 2$ and $x = 3$. We will leave our answer in this form. To find the horizontal asymptote let's look at our original equation. The highest power on the top is the same as the highest power on the bottom, so we use rule 2 again. Our $a_n = -1$ and $b_m = 1$, so the horizontal asymptote is $y = \frac{-1}{1}$ so y = -1. The domain is all numbers except $x = 2$ and $x = 3$. In interval notation, this is $(-\infty,2)\cup(2,3)\cup(3,\infty)$. Notice the domain coincides with the vertical asymptotes.

EXAMPLE: Find the intercepts and asymptotes but DO NOT GRAPH: $y = \dfrac{x^2 - 3x}{2x^2 + 4x^3}$.

First you want to factor to see if it can be simplified further: $y = \dfrac{x(x-3)}{2x^2(1+2x)} = \dfrac{x-3}{2x(1+2x)}$. You always want to simplify if possible. So now let's find the y-intercept. To do this we need to put in a 0 for x. When you do you will get a zero in the denominator, which is undefined. So there is no y-intercept. To find the x-intercept we need to set the top equal to zero. So we have $x-3=0$. Solving this we get x = 3, so this is (3, 0).

Now let's find the asymptotes. To find the vertical asymptote we need to set the bottom equal to zero. So we have $2x(1+2x)=0$ so $x=0$ and $x = -\frac{1}{2}$ We will leave our answer in this form. To find the horizontal asymptote let's look at our original equation. The highest power on the top is less than the highest power on the bottom, so we use rule 1. This says that the horizontal asymptote is automatically y = 0.

EXAMPLE: Find the asymptotes but DO NOT GRAPH: $y = \dfrac{3x^2 + 10x + 6}{x+2}$

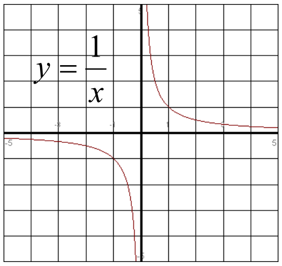
This does not ask us to find intercepts, so we will just find the asymptotes. To find the vertical asymptote we need to set the bottom equal to zero. So we have $x+2=0$ so $x=-2$. We will leave our answer in this form. To find the horizontal asymptote let's look at our original equation. The highest power on the top is more than the highest power on the bottom, so now we have rule 3. This tells us there is no horizontal asymptote, but there is an oblique asymptote. We need to find this. In order to do that we must use long division like we did in a previous section

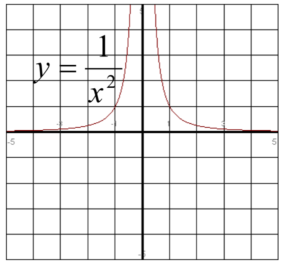
$$\begin{array}{r} 3x+4 \\ x+2\overline{)\,3x^2+10x+6} \\ \underline{3x^2+6x} \\ 4x+6 \\ \underline{4x+8} \\ -2 \end{array}$$

From doing the long division we get 3x + 4. This is the equation of the oblique asymptote. We always ignore the remainder. We just write y = 3x + 4.

More on next page…

Next, we will look at some special graphs. These are: $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$. The normal graphs look like:

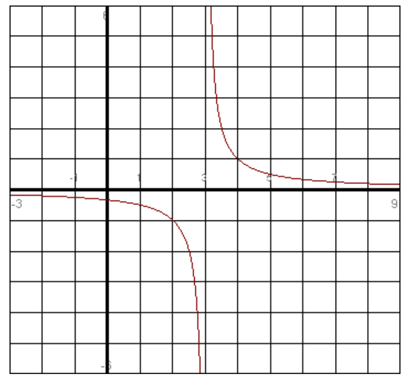




We can still graph by using transformations like we did in previous sections.

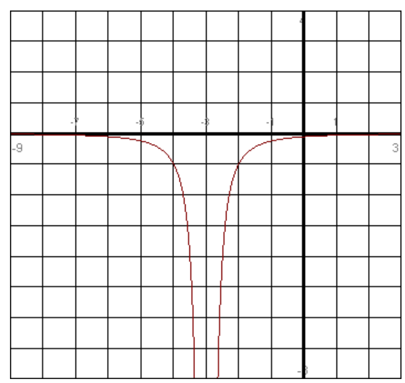
EXAMPLE: Graph $y = \frac{1}{x-3}$ using transformations.

The x – 3 says we need to move the graph of $y = \frac{1}{x}$ three places to the right. We will get:



EXAMPLE: Graph $y = \frac{-1}{x^2 + 6x + 9}$ using transformations.

First we will factor: $y = \frac{-1}{(x+3)(x+3)}$ which can be written as $y = \frac{-1}{(x+3)^2}$. The x + 3 tells us that we need to move $y = \frac{1}{x^2}$ three places to the left. The negative tells us we need to flip the graph horizontally.



EXAMPLE: Graph $y = \frac{x-1}{x}$ using transformations.

First I will divide everything by x to get $y = \frac{x}{x} - \frac{1}{x}$ or $y = -\frac{1}{x} + 1$

This tells us to move the graph of $y = \frac{1}{x}$ up one unit and then flip it horizontally.