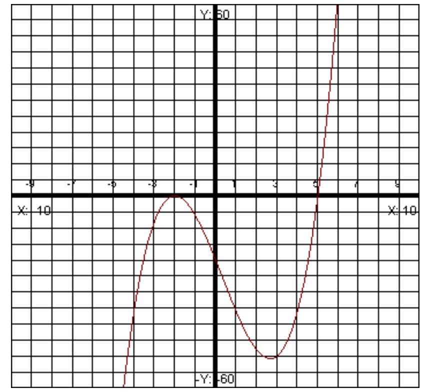
4.5 Polynomial and Rational Inequalities

This section is looking a sections of graphs that are above and below the x-axis.

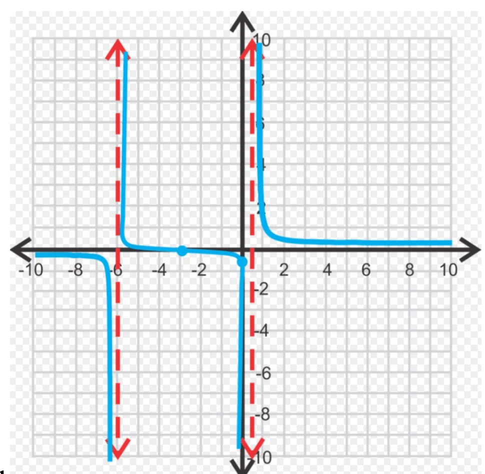
EXAMPLE: Use the graph of $f(x)$ given below to solve: (a) $f(x) \geq 0$ and (b) $f(x) < 0$.



(a) If $f(x) \geq 0$, this means that we are looking for where the graph is above the x-axis. This only occurs for x-values greater than 5, so our answer in interval notation is $[5, \infty)$. We are using a bracket on the 5 since we have the $\geq$ symbol here. 5 is included.

(b) If $f(x) < 0$, this means that we are looking for where the graph is below the x-axis. This occurs from negative infinity to 5. At the x-value of –2, the graph touches the x-axis, so the y-value here is zero. This is not included in our answer because if $f(x)$ is zero we will have 0 < 0 which is not a true statement. Therefore zero is not included in our answer. In interval notation we will write: $(-\infty, -2) \cup (-2, 5)$.

EXAMPLE: Use the graph of $f(x)$ given below to solve: (a) $f(x) > 0$ and (b) $f(x) \leq 0$.



(a) If $f(x) > 0$, this means that we are looking for where the graph is above the x-axis. This occurs for x-values between –6 and–3 and for numbers greater than $\frac{1}{2}$. We will write $(-6, -3) \cup \left(\frac{1}{2}, \infty\right)$. Note the use of all parenthesis here since we have the > sign and the endpoints are not included.

(b) If $f(x) \leq 0$, this means that we are looking for where the graph is below the x-axis. This occurs for x-values less than –6 and between –3 and $\frac{1}{2}$.

We will write $(-\infty, -6) \cup \left[-3, \frac{1}{2}\right)$. We see that the –3 is included so we need a bracket. However the other numbers need parenthesis since the graph is undefined at these numbers.

We have seen how to find the answers to these inequalities by using a graph. However, what happens if a graph is not included? We need a process to find this algebraically which is what we will look at next.

EXAMPLE: Solve and write you answer in interval notation: $(x-5)(x+2)^2 < 0$

We can rewrite the above equation as $(x-5)(x+2)(x+2) < 0$. We need to find the critical points. This is what makes each factor equal to zero. This would be x = 5 and x = -2. Now we want to set up a table with our critical points and factors. Even though x + 2 is repeated we still need to include BOTH of them on our table:

x – 5			
x + 2			
x + 2			

-2 5

So our table is complete with the factors down the left column and our critical point in order from smallest to largest along the bottom. Now we need to put in our test values. You need to test a value less than -2, and this can be any number less than -2. I will use -3. In the middle column I need to pick any number between -2 and 5, so I will use 0. In the last column I need to test a value that is larger than 5, so I will use 6. Put these test values in for x in all three factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. For example when I test -3 I will put this in for x in the expression x – 5 and I will get -3 – 5, which is -8. Since this is negative I will put a negative sign in that part of the table. Do this for all the other values and you will get this table:

x – 5	-	-	+
x + 2	-	+	+
x + 2	-	+	+

-2 5

Now is the step where I need to multiply the signs together in each column. In the left column I have three negatives, so I will multiply those together to get a negative. In the middle column I have one negative and two positives so when I multiply them I will get a negative. The last column has three positives so when I multiply them I get a positive. I want to put these results on the table:

x – 5	-	-	+
x + 2	-	+	+
x + 2	-	+	+

- -2 - 5 +

The factored form was $(x-5)(x+2)(x+2) < 0$ which means that whatever we get for x must be less than zero. This means we are looking for negatives. Since we want negative values, we want to indicate the regions on our table that resulted in being negative. We notice that everything to the left of -2 was negative and everything between -2 and 5 was negative. We will write $(-\infty,-2)\cup(-2,5)$ as our answer. Are you wondering if we can write our answer as $(-\infty,5)$? The answer is no. The reason why is because when x = -2 the result will be zero. In our problem it ends with < 0, meaning that zero is NOT included, so since -2 gives a zero it can't be included.

Notice we got the same answer as in our first example. We did the same problem algebraically.

EXAMPLE: Solve and write you answer in interval notation: $6x^2 < 6 + 5x$

We want to rewrite the above equation as $6x^2 - 5x - 6 < 0$ so that a zero is on the right hand side. We need to find the critical points. This is what makes each factor equal to zero. If we factor this we will get: $(3x+2)(2x-3) < 0$. Setting it equal to zero we get x = -2/3 and x = 3/2. Now we want to set up a table :

$3x+2$			
$2x-3$			

-2/3 3/2

So our table is complete with the factors down the left column and our critical point in order from smallest to largest along the bottom. Now we need to put in our test values. You need to test a value less than -2/3, and this can be any number less than -2/3. I will use -1. In the middle column I need to pick any number between -2/3 and 3/2, so I will use 0. In the last column I need to test a value that is larger than 3/2, so I will use 2. Put these test values in for x in the two factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. For example when I test -1 I will put this in for x in the expression $3x+2$ and I will get $3(-1)+2 = -1$. Since this is negative I will put a negative sign in that part of the table. Do this for all the other values and you will get this table:

$3x+2$	-	+	+
$2x-3$	-	-	+

-2/3 3/2

Now is the step where I need to multiply the signs together in each column. In the left column I have two negatives, so I will multiply those together to get a positive. In the middle column I have a positive and a negative, so when I multiply them I will get a negative. The last column has two positives so when I multiply them I get a positive. I want to put these results on the table:

$3x+2$	-	+	+
$2x-3$	-	-	+

\+ -2/3 - 3/2 +

The factored form was $(3x+2)(2x-3) < 0$ which means that whatever we get for x must be less than zero. This means we are looking for negatives. Since we want negative values, we want to indicate the regions on our table that resulted in being negative. We notice that everything between -2/3 and 3/2 was negative. We will write $\left(-\frac{2}{3}, \frac{3}{2}\right)$ as our answer.

More on next page…

EXAMPLE: Solve and write you answer in interval notation: $3x^3 \geq -15x^2$

We want to rewrite the above equation as $3x^3 + 15x^2 \geq 0$ so that a zero is on the right hand side. We need to find the critical points. This is what makes each factor equal to zero. If we factor this we will get: $3x^2(x+5) \geq 0$. Setting it equal to zero we get x = 0 and x = -5. Now we want to set up a table :

$3x^2$			
x + 5			

-5 0

So our table is complete with the factors down the left column and our critical point in order from smallest to largest along the bottom. Now we need to put in our test values. You need to test a value less than -5, and this can be any number less than -5. I will use -6. In the middle column I need to pick any number between -5 and 0, so I will use -1. In the last column I need to test a value that is larger than 0, so I will use 1. Put these test values in for x in the two factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. For example when I test -6 I will put this in for x in the expression $3x^2$ and I will get $3(-6)^2 = 108$. Since this is positive I will put a positive sign in that part of the table. Do this for all the other values and you will get this table:

$3x^2$	+	+	+
x + 5	-	+	+

-5 0

Now is the step where I need to multiply the signs together in each column. In the left column I have a positive and a negative, so I will multiply those together to get a negative. In the middle column I have two positives so when I multiply them I will get a positive. The last column has two positives so when I multiply them I get a positive. I want to put these results on the table:

$3x^2$	+	+	+
x + 5	-	+	+

- -5 + 0 +

The factored form was $3x^2(x+5) \geq 0$ which means that whatever we get for x must be greater than or equal to zero. This means we are looking for positives. Since we want positive values, we want to indicate the regions on our table that resulted in being positive. We notice that everything between -5 and 0 was positive and everything to the right of zero is positive. This time 0 can be included since the sign was ≥ 0. We will write $[-5, \infty)$ as our answer. Are you wondering if we can write our answer as $[-5,0) \cup (0, \infty)$? The answer is no. The reason why is because 0 should be included this time because we have ≥ 0.

EXAMPLE: Solve and write you answer in interval notation: $\frac{(x-2)(x+1)}{x-4} \leq 0$

This problem is already factored for us. For these rational functions we want to also find the critical points. To do this you will set all factors in the numerator and denominator equal to zero. You will get x = 2, -1 and 4 as the critical points. Now we will set up a table:

x – 2				
x + 1				
x – 4				

-1 2 4

So our table is complete with the factors from both top and bottom of the fraction down the left column and our critical points are in order from smallest to largest along the bottom. Now we need to put in our test values. In the first column I need to pick a number less than -1, so I will use -2. In the second column I need to pick any number between -1 and 2, so I will use 0. In the third column I need to test a value that is between 2 and 4, so I will use 3. In the fourth column I need to test a value that is greater than 4, so I will use 5. Put these test values in for x in all three factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. After you finish that your table should look like:

x – 2	-	-	+	+
x + 1	-	+	+	+
x – 4	-	-	-	+

-1 2 4

Now is the step where I need to multiply the signs together in each column. In the first column I have three negatives, so I will multiply those together to get a negative. In the second column I have two negatives and a positive so when I multiply them I will get a positive. The third column has two positives and a negative so when I multiply them I get a negative. The fourth column has three positives, so when I multiply those I get a positive. I want to put these results on the table:

x – 2	-	-	+	+
x + 1	-	+	+	+
x – 4	-	-	-	+

\- -1 + 2 - 4 +

The factored form was $\frac{(x-2)(x+1)}{x-4} \leq 0$ which means that whatever we get for x must be less than or equal to zero. This means we are looking for negatives. Since we want negative values, we want to indicate the regions on our table that resulted in being negative. Our negative regions include everything less than -1 and also everything between 2 and 4. So we write our answer as: $(-\infty,-1]\cup[2,4]$. **BUT WAIT!** Do you see an error in my answer? I am supposed to have brackets in my answer, but I do not want them around the 4 because this makes the denominator equal to zero and we can't divide by zero. Therefore I do not want to include the 4 even though my original problem had brackets. My final correct answer is: $(-\infty,-1]\cup[2,4)$.

EXAMPLE: Solve and write you answer in interval notation: $\frac{x^2-4}{x-3} > 0$

First we should factor this as: $\frac{(x-2)(x+2)}{x-3} > 0$.

We need to find our critical points, where we set each factor equal to zero no matter if it is on the top or bottom. You will get x = 2, x = -2, and x = 3. These are the critical values that we will put on our table:

x + 2				
x – 2				
x – 3				

-2 2 3

So our table is complete with the factors from both top and bottom of the fraction down the left column and our critical points are in order from smallest to largest along the bottom. Now we need to put in our test values. In the first column I need to pick a number less than -2, so I will use -3. In the second column I need to pick any number between -2 and 2, so I will use 0. In the third column I need to test a value that is between 2 and 3, so I will use 2.5. In the fourth column I need to test a value that is greater than 3, so I will use 4. Put these test values in for x in all three factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. After you finish that your table should look like:

x + 2	-	+	+	+
x – 2	-	-	+	+
x – 3	-	-	-	+

-2 2 3

Now is the step where I need to multiply the signs together in each column. In the first column I have three negatives, so I will multiply those together to get a negative. In the second column I have two negatives and a positive so when I multiply them I will get a positive. The third column has two positives and a negative so when I multiply them I get a negative. The fourth column has three positives, so when I multiply those I get a positive. I want to put these results on the table:

x + 2	-	+	+	+
x – 2	-	-	+	+
x – 3	-	-	-	+

- -2 + 2 - 3 +

The factored form was $\frac{(x-2)(x+2)}{x-3} > 0$ which means that whatever we get for x must be greater than zero. This means we are looking for positives. Since we want positive values, we want to indicate the regions on our table that resulted in being positive. Our positive regions include everything between -2 and 2 and also everything greater than 3. So we write our answer as: $(-2,2) \cup (3,\infty)$. We don't need to worry about dividing by zero like the previous problem. Three is not included anyway.

EXAMPLE: Solve and write you answer in interval notation: $\frac{x+2}{x-4} > 1$

The problem with this one is that there is not a zero on the right hand side. The process we have been doing only works if there is zero on the right side. So we can subtract 1 from both sides to get: $\frac{x+2}{x-4} - 1 > 0$. But now we need a single fraction on the left side so we need to get common denominators: $\frac{x+2}{x-4} - 1\left(\frac{x-4}{x-4}\right) > 0$. Combining to one fraction we get: $\frac{x+2-(x-4)}{x-4} > 0$. Distribute the negative to get: $\frac{x+2-x+4}{x-4} > 0$.

After simplifying we will get $\frac{6}{x-4} > 0$. Now for this one we will only have one critical value, which is x = 4. The top of the fraction can't be set equal to zero, which is why we only have one critical point here.

6		
x – 4		
	4	

So all we need to do now is test a value less than 4 and greater than 4. I will test 0 and 5. The top row of the table will automatically by positive since 6 is positive. There is no variable to put a test value in on the top. On the bottom row I can put in my test value. Let's complete the table:

6	+	+
x – 4	-	+
	4	

Now is the step where I need to multiply the signs together in each column. In the left column I have a positive and a negative, so I will multiply those together to get a negative. In the right column I have two positives so when I multiply them I will get a positive. I want to put these results on the table:

6	+	+
x – 4	-	+
	- 4	+

The factored form was $\frac{6}{x-4} > 0$ which means that whatever we get for x must be greater than zero. This means we are looking for positives. Since we want positive values, we want to indicate the regions on our table that resulted in being positive. We notice that everything greater than 4 was positive. This is the only region that was positive so we write our answer as $(4, \infty)$.

EXAMPLE: Solve and write you answer in interval notation: $\dfrac{2x^2+3x-4}{x+1} \le 2$

This is another one where we need to get a zero on the right hand side. We first subtract 2 from both sides to get: $\dfrac{2x^2+3x-4}{x+1} - 2 \le 0$. Now we need to get common denominators again: $\dfrac{2x^2+3x-4}{x+1} - 2\left(\dfrac{x+1}{x+1}\right) \le 0$.

Combining to one fraction we get: $\dfrac{2x^2+3x-4-2(x+1)}{x+1} \le 0$. Now distribute the -2 and you will get:

$\dfrac{2x^2+3x-4-2x-2}{x+1} \le 0$. After simplifying you should get: $\dfrac{2x^2+x-6}{x+1} \le 0$. Now you can factor the top using any method. You will get: $\dfrac{(x+2)(2x-3)}{x+1} \le 0$. Now we will find the critical points now that it is factored. You will get x = -2, $\dfrac{3}{2}$, and -1. Now we can make our table:

x + 2				
2x – 3				
x + 1				

-2 -1 3/2

So our table is complete with the factors from both top and bottom of the fraction down the left column and our critical points are in order from smallest to largest along the bottom. Now we need to put in our test values. In the first column I need to pick a number less than -2, so I will use -3. In the second column I need to pick any number between -2 and -1, so I will use -1.5. In the third column I need to test a value that is between -1 and 3/2, so I will use 0. In the fourth column I need to test a value that is greater than 3/2, so I will use 2. Put these test values in for x in all three factors in the left column and put a negative sign if you get a negative number and a positive if the number is positive. After you finish that your table should look like:

x + 2	-	+	+	+
2x – 3	-	-	-	+
x + 1	-	-	+	+

-2 -1 3/2

Now is the step where I need to multiply the signs together in each column. In the first column I have three negatives, so I will multiply those together to get a negative. In the second column I have two negatives and a positive so when I multiply them I will get a positive. The third column has two positives and a negative so when I multiply them I get a negative. The fourth column has three positives, so when I multiply those I get a positive. I want to put these results on the table:

x + 2	-	+	+	+
2x – 3	-	-	-	+
x + 1	-	-	+	+

\- -2 + -1 - 3/2 +

The factored form was $\frac{(x+2)(2x-3)}{x+1} \leq 0$ which means that whatever we get for x must be less than or equal to zero. This means we are looking for negatives. Since we want negative values, we want to indicate the regions on our table that resulted in being negative. Our negative regions include everything less than -2 and between -1 and 3/2. So we write our answer as: $(-\infty,-2] \cup \left(-1,\frac{3}{2}\right]$. Notice that -1 has a parenthesis because this will cause the bottom of the fraction to be zero.