5.1 Evaluate Composite Functions

Sum, Difference, Product, and Quotient of Functions

Given functions f and g, the functions $f+g$, $f-g$, $f\cdot g$, and $\dfrac{f}{g}$ are defined by

$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(f\cdot g)(x) = f(x)\cdot g(x)$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided that $g(x) \neq 0$

EXAMPLE: Let $f(x) = x^2 - 9x$, $g(x) = -6x$.

a.) Find $(f+g)(x)$ and $(f+g)(-2)$

So we use the definition: $(f+g)(x) = f(x) + g(x)$ and substitute in our functions. Therefore, $(f+g)(x) = x^2 - 9x + -6x$. This simplifies to $(f+g)(x) = x^2 - 15x$.

$(f+g)(-2) = (-2)^2 - 15(-2) = 34$

b.) Find $(f-g)(x)$ and $(f-g)(1)$

So we use the definition: $(f-g)(x) = f(x) - g(x)$ and substitute in our functions. Therefore $(f-g)(x) = x^2 - 9x - -6x$. This simplifies to $(f+g)(x) = x^2 - 3x$.

$(f+g)(1) = 1^2 - 3(1) = -2$

c.) Find $(f\cdot g)(x)$ and $(f\cdot g)(3)$

So we use the definition: $(f\cdot g)(x) = f(x)\cdot g(x)$ and substitute in our functions. Therefore $(f\cdot g)(x) = (x^2 - 9x)\cdot -6x$. This simplifies to $(f+g)(x) = -6x^3 + 54x$.

$(f+g)(3) = -6(3)^3 + 54(3) = -6(27) + 162 = 0$

d.) Find $\left(\dfrac{f}{g}\right)(x)$ and $\left(\dfrac{f}{g}\right)(15)$

So we use the definition: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ and substitute in our functions. Therefore $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2 - 9x}{-6x}$. This simplifies to $\left(\dfrac{f}{g}\right)(x) = \dfrac{x-9}{-6}$. $\left(\dfrac{f}{g}\right)(15) = \dfrac{15-9}{-6} = \dfrac{6}{-6} = -1$

Composite Functions – a way of combining two functions

$(f \circ g)(x) = f(g(x))$ This is pronounced "f of g of x" DOES NOT MEAN F TIMES G!!!
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These are not multiplications. The $(f \circ g)(x)$ means we place the g function into the f function.
The $(g \circ f)(x)$ means we place the f function into the g function.

EXAMPLE: Given: $f(x) = 5x - 4$ and $g(x) = 3x + 1$ find the following:
$(f \circ g)(2)$, $(g \circ f)(-1)$, $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$, $(g \circ g)(x)$

a.) $(f \circ g)(2) = f(g(2))$ First we rewrite this using the definition, replacing x with 2.

Let's first do $g(2)$ since this is inside the parenthesis. To do this, replace the x with 2 in the g function:
$g(2) = 3(2) + 1 = 7$ Since now we know that $g(2) = 7$ we can now replace $g(2)$ with 7:
By replacing $g(2)$ with 7 now we need to find $f(7)$. Replace the x in the f function with 7:
$f(7) = 5(7) - 4 = 31$. So now conclude that $(f \circ g)(2) = 31$.

b.) $(g \circ f)(-1) = g(f(-1))$ First we rewrite this using the definition, replacing x with -1.

Let's first do $f(-1)$ since this is inside the parenthesis. To do this, replace the x with -1 in the f function:
$f(-1) = 5(-1) - 4 = -9$ Since now we know that $f(-1) = -9$ we can now replace $f(-1)$ with -9:
By replacing $f(-1)$ with -9 now we need to find $g(-9)$. Replace the x in the g function with -9:
$g(-9) = 3(-9) + 1 = -26$. So now conclude that $(g \circ f)(-1) = -26$.

c.) $(f \circ g)(x) = f(g(x))$ We use the definition.

Since we don't have a number for x this time there will not be a number answer for this one. What we can do is replace the $g(x)$ with the expression $3x + 1$. So now we will do $f(3x + 1)$. This means where ever there is an x in the f function we will replace it with 3x + 1:
$f(3x + 1) = 5(3x + 1) - 4$ Now simplify:
$f(3x + 1) = 15x + 5 - 4 = 15x + 1$ So we write our answer as: $(f \circ g)(x) = 15x + 1$
Notice what happens if we put a 2 in for x in our answer. We will get 31, which is the same as in part a.)

d.) $(g \circ f)(x) = g(f(x))$ We use the definition.

Since we don't have a number for x this time there will not be a number answer for this one. What we can do is replace the $f(x)$ with the expression $5x - 4$. So now we will do $g(5x - 4)$. This means where ever there is an x in the g function we will replace it with 5x - 4:
$g(5x - 4) = 3(5x - 4) + 1$ Now simplify:
$g(5x - 4) = 15x - 12 + 1 = 15x - 11$ So we write our answer as: $(g \circ f)(x) = 15x - 11$
Notice what happens if we put a -1 in for x in our answer. We will get -26, which is the same as in part b.)

e.) $(f \circ f)(x) = f(f(x))$

So for this one we are putting in f into itself. Replace $f(x)$ inside with $5x - 4$. You will have $f(5x-4)$. Now replace the x in $5x - 4$ with $5x - 4$: $5(5x-4)-4$. Now simplify: $25x - 20 - 4 = 25x - 24$. Therefore $(f \circ f)(x) = 25x - 24$.

f.) $(g \circ g)(x) = g(g(x))$

So for this one we are putting in g into itself. Replace $g(x)$ inside with $3x + 1$. You will have $g(3x+1)$. Now replace the x in $3x + 1$ with $3x + 1$: $3(3x+1)+1$. Now simplify: $9x+3+1 = 9x+4$. Therefore $(g \circ g)(x) = 9x + 4$.

EXAMPLE: Given: $f(x) = x + 3$ and $g(x) = 2x^2 - 1$ find the following:
$(f \circ g)(-1)$, $(g \circ f)(0)$, $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$, $(g \circ g)(x)$

For this one the process is the same as I described about. I will only show the algebraic steps here.

a.) $(f \circ g)(-1) = f(g(-1))$
$g(-1) = 2(-1)^2 - 1$
$g(-1) = 1$
$f(g(-1)) = f(1) = 1 + 3 = 4$, so $(f \circ g)(-1) = 4$

b.) $(g \circ f)(0) = g(f(0))$
$f(0) = 0 + 3$
$f(0) = 3$
$g(f(0)) = g(3) = 2(3)^2 - 1 = 17$, so $(g \circ f)(0) = 17$

c.) $(f \circ g)(x) = f(g(x))$
$f(2x^2 - 1) = (2x^2 - 1) + 3$, so $(f \circ g)(x) = 2x^2 + 2$

d.) $(g \circ f)(x) = g(f(x))$
$g(x+3) = 2(x+3)^2 - 1 = 2(x^2 + 6x + 9) - 1 = 2x^2 + 12x + 18 - 1$, so $(g \circ f)(x) = 2x^2 + 12x + 17$

e.) $(f \circ f)(x) = f(f(x))$
$f(x+3) = (x+3) + 3 = x + 6$, so $(f \circ f)(x) = x + 6$

f.) $(g \circ g)(x) = g(g(x))$
$g(2x^2 - 1) = 2(2x^2 - 1)^2 - 1 = 2(4x^4 - 4x^2 + 1) - 1 = 8x^4 - 8x^2 + 2 - 1$
$(g \circ g)(x) = 8x^4 - 8x^2 + 1$

EXAMPLE: Given: $f(x)=4-x^2$ and $g(x)=\dfrac{-1}{x}$ find the following: $(f\circ g)(0)$, $(f\circ g)(x)$, $(g\circ f)(x)$, $(f\circ f)(x)$, $(g\circ g)(x)$. Find the domain of $(f\circ g)(x)$, $(g\circ f)(x)$, $(f\circ f)(x)$, $(g\circ g)(x)$ in interval notation.

a.) $(f\circ g)(0)=f(g(0))$
$g(0)=undefined$
Since this is undefined, that means that $(f\circ g)(0)$ is also undefined.

b.) $(f\circ g)(x)=f(g(x))$

$$f\left(\frac{-1}{x}\right)=4-\left(\frac{-1}{x}\right)^2=4-\frac{1}{x^2}$$

$$(f\circ g)(x)=4-\frac{1}{x^2}$$

Now we want to find the domain. 0 is not allowed since we cannot divide by zero. Therefore, the domain is $(-\infty,0)\cup(0,\infty)$.

c.) $(g\circ f)(x)=g(f(x))$

$g(4-x^2)=\dfrac{-1}{4-x^2}$ We can factor the denominator to help with our domain.

$$(g\circ f)(x)=\frac{-1}{(2-x)(2+x)}$$

The numbers 2 and -2 are not allowed since they cause division by zero. Therefore, the domain is $(-\infty,-2)\cup(-2,2)\cup(0,\infty)$.

d.) $(f\circ f)(x)$

$$f(f(x))=f(4-x^2)=4-(4-x^2))^2=4-(x^4-8x^2+16)=-x^4+8x^2-12$$

Since this is a polynomial, the domain would include all real numbers. Nothing causes division by zero. Therefore the domain is: $(-\infty,\infty)$.

e.) $(g\circ g)(x)$

$$g(g(x))=g\left(-\frac{1}{x}\right)$$

$$g(g(x))=\frac{-1}{\left(-\frac{1}{x}\right)}$$

$$g(g(x)=-1\left(-\frac{x}{1}\right)=x$$

Even though the answer here was x, we need to be careful on this one. We know we can put in any number for x, however if you look at the algebra in the second line, we have $g(g(x)=\dfrac{-1}{\left(-\dfrac{1}{x}\right)}$. At this point we see that x cannot be 0 otherwise we will be dividing by zero. Therefore, the domain for this problem is $(-\infty,0)\cup(0,\infty)$. Therefore, when finding domains on these fraction problems, be careful. Make sure you consider all the algebraic steps. This also applies on our next problem:

EXAMPLE: If $f(x)=\dfrac{5}{x-3}$ and $g(x)=\dfrac{1}{x}$, find the domain of $(f\circ g)(x)$ and $(g\circ f)(x)$ using interval notation.

First, we will find $(f\circ g)(x)=f(g(x))=f\left(\dfrac{1}{x}\right)$. You will get $\dfrac{5}{\dfrac{1}{x}-3}$. Before we simplify this one, let's first analyze what we have. First of all we notice that if we put a 0 into this expression, it will be undefined. Therefore we know right away that 0 is not in the domain. Now let's keep going and simplify this as far as we can go: $(f\circ g)(x)=\dfrac{5}{\dfrac{1}{x}-3}=\dfrac{5}{\dfrac{1-3x}{x}}=\dfrac{5x}{1-3x}$. So now let's find the domain of $(f\circ g)(x)=\dfrac{5x}{1-3x}$. This means the bottom cannot equal zero. Setting the bottom equal to zero we get that x cannot be 1/3. So the domain is $(-\infty,0)\cup\left(0,\dfrac{1}{3}\right)\cup\left(\dfrac{1}{3},\infty\right)$.

Next, we will find $(g\circ f)(x)=g(f(x))=g\left(\dfrac{5}{x-3}\right)$. You will get $\dfrac{1}{\left(\dfrac{5}{x-3}\right)}$. Before we simplify this one, let's first analyze what we have. First of all we notice that if we put a 3 into this expression, it will be undefined. Therefore, we know right away that 3 is not in the domain. Now let's keep going and simplify this as far as we can go: $(g\circ f)(x)=\dfrac{1}{\left(\dfrac{5}{x-3}\right)}=\dfrac{x-3}{5}$. This time, nothing causes the denominator to be 0. So the only number that is not allowed in the domain is 3. The domain is $(-\infty,3)\cup(3,\infty)$.

EXAMPLE: Find functions f and g so that $(f\circ g)(x)=H(x)$ given that $H(x)=(1+x^2)^3$

This problem is the reverse of what we were just doing. Now we have the finished product and we want to go back to the beginning. We want to find the two functions f and g so that when we find $(f\circ g)(x)$ we will get the expression $(1+x^2)^3$. First let's find the $g(x)$. This is usually an 'inside' function, one put into something else. In H we have the expression $1+x^2$. This is inside a set of parenthesis raised to the third power. So we will let $g(x)=1+x^2$. Now the 'outside' function is the $f(x)$. In order to find this function, replace the 'inside' function with x in H(x) and then replace H(x) with f(x). In this case we will replace $1+x^2$ with x. You will get $f(x)=x^3$.

Now let's check to make sure we did it correctly. If we let $f(x) = 1 + x^2$ and $g(x) = x^3$ let's find $(f \circ g)(x)$. To do this, we place the g into the f. We will get $(1+x^2)^3$, which is equal to H. So the answers are: $f(x) = x^3$ and $g(x) = 1 + x^2$.

EXAMPLE: Find functions f and g so that $(f \circ g)(x) = H(x)$ given that $H(x) = \sqrt{2x-1} - 4x + 2$

First let's factor H: $H(x) = \sqrt{2x-1} - 2(2x-1)$. Notice we have the expression 2x – 1 repeating. This is our 'inside' function, so $g(x) = 2x - 1$. To find $f(x)$, replace the 2x – 1 with x in the H(x) equation and then replace H(x) with *f*(*x*). You will get $f(x) = \sqrt{x} - 2x$. So our answers are $g(x) = 2x - 1$ and $f(x) = \sqrt{x} - 2x$.

EXAMPLE: Use the given table to evaluate each composition.

x	–3	–2	–1	0	1	2	3
$f(x)$	11	9	7	5	3	1	–1
$g(x)$	–8	–3	0	1	0	–3	–8

a.) Find $(f \circ g)(1)$.

First we can rewrite this as $f(g(1))$. First we will find $g(1)$. Go to the x row and find 1. Then go down to the $g(x)$ row. Therefore $g(1) = 0$. So now our problem becomes $f(0)$. To find this we will go to the x row and find 0. Then go down to the $f(x)$ row and you will find $f(0) = 5$. Therefore, $(f \circ g)(1) = 5$.

b.) Find $(f \circ g)(2)$.

First we can rewrite this as $f(g(2))$. First we will find $g(2)$. Go to the x row and find 2. Then go down to the $g(x)$ row. Therefore $g(2) = -3$. So now our problem becomes $f(-3)$. To find this we will go to the x row and find –3. Then go down to the $f(x)$ row and you will find $f(-3) = 11$. Therefore, $(f \circ g)(2) = 11$.

c.) Find $(g \circ f)(2)$.

First we can rewrite this as $g(f(2))$. First we will find $f(2)$. Go to the x row and find 2. Then go down to the $f(x)$ row. Therefore $f(2) = 1$. So now our problem becomes $g(1)$. To find this we will go to the x row and find 1. Then go down to the $g(x)$ row and you will find $g(1) = 0$. Therefore, $(g \circ f)(2) = 0$.

d.) Find $(g \circ g)(1)$.

First we can rewrite this as $g(g(1))$. First we will find $g(1)$. Go to the x row and find 1. Then go down to the $g(x)$ row. Therefore $g(1) = 0$. So now our problem becomes $g(0)$. To find this we will go to the x row and find 0. Then go down to the $g(x)$ row and you will find $g(0) = 1$. Therefore, $(g \circ g)(1) = 1$.

e.) Find $(f \circ f)(3)$.

First we can rewrite this as $f(f(3))$. First we will find $f(3)$. Go to the x row and find 3. Then go down to the $f(x)$ row. Therefore $f(3) = -1$. So now our problem becomes $f(-1)$. To find this we will go to the x row and find –1. Then go down to the $f(x)$ row and you will find $f(-1) = 7$. Therefore, $(f \circ f)(3) = 7$.