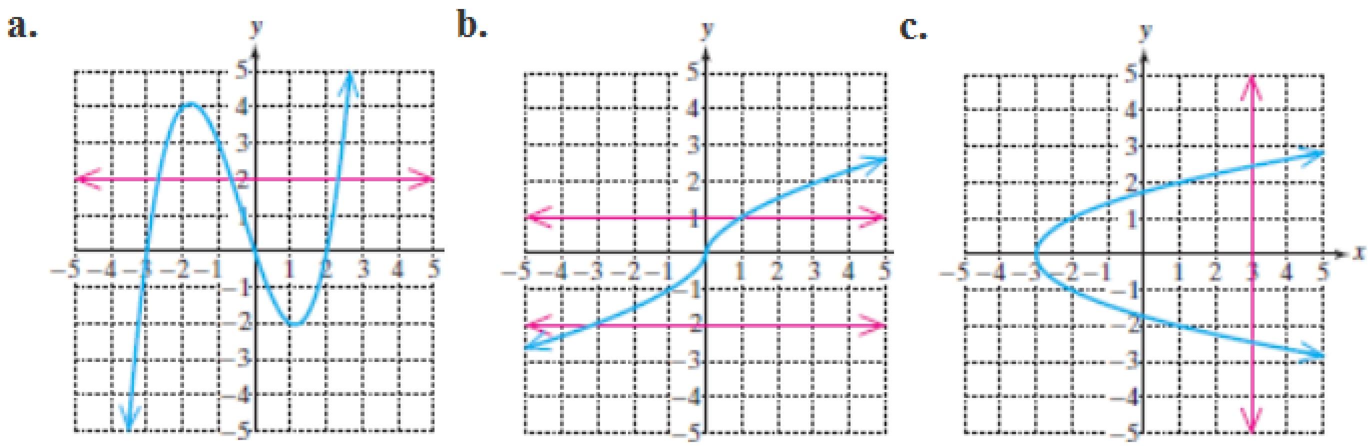
5.2 One to One Functions; Inverse Functions

One-To-One Function

A function *f* is a one-to-one function if for a and b in the domain of *f*, if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$ then $a = b$. In other words, for each y value there can only be one x value.

Horizontal Line Test

If you pass a horizontal line and it hits the graph at only one place, then it is one-to-one.
EXAMPLE: Use the horizontal line test to determine if the graph below defines y as a one-to-one function.



For part **a**, the horizontal line hits the graph three times. Therefore it is not one-to-one.
For part **b**, the horizontal line hits the graph only once no matter where it is drawn. Therefore it is one-to-one.
For part **c**, it does not pass the vertical line test. Therefore it is not a function, and not one-to-one.

Inverse Functions

Let *f* be a one-to-one function. Then *g* is the inverse of *f* if the following conditions are both true:
$(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

EXAMPLE: Given $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}x + \frac{1}{2}$ verify that they are inverses.

We need to show $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. Let's first find $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$ Start with the definition.

$(f \circ g)(x) = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ We remove the g(x) and replace it with $\frac{1}{2}x + \frac{1}{2}$.

$(f \circ g)(x) = 2\left(\frac{1}{2}x + \frac{1}{2}\right) - 1$ Now simplify.

$(f \circ g)(x) = x + 1 - 1$

$(f \circ g)(x) = x$ We have shown this is true. Now we need to show $(g \circ f)(x) = x$.

$(g \circ f)(x) = g(f(x))$ First start with the definition.

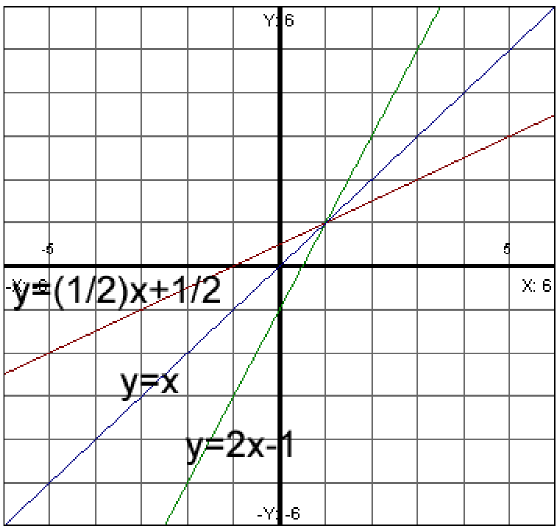
$(g \circ f)(x) = g(2x-1)$ We remove the f(x) and replace it with 2x – 1.

$(g \circ f)(x) = \frac{1}{2}(2x-1) + \frac{1}{2}$ Now simplify.

$(g \circ f)(x) = x - \frac{1}{2} + \frac{1}{2}$

$(g \circ f)(x) = x$ We have shown this is true. So we have verified they are inverses.

What is the significance of the *x*? Why do we get *x* when we simplify? I'm glad you asked! Let's look at the graph of *f* and *g*. I will also graph *y* = *x*. Notice that f and g are symmetric to the line *y* = *x*. This is always the case with inverses. Notice also that points on the graph of *f*(*x*) are reversed on g(x). For example, on the *f*(*x*) line we see the points (2, 3) and (-1, -3). On the graph of *g*(*x*) we get the points (3, 2) and (-3, -1).



It is not a coincidence that the points from *f*(*x*) are reversed on *g*(*x*). It just so happens that this is the way to find inverses algebraically.

Notation to write "the inverse of *f*(*x*)" is $f^{-1}(x)$. This does not mean *f* raised to the negative one power. It just means we have the inverse of *f*(*x*).

We just talked about the fact that the x and y coordinates are reversed on the graph. This means the following: If $f(x) = y$, then $f^{-1}(y) = x$.

EXAMPLE: If you are given $f(10) = 14$, find $f^{-1}(14)$.

We can still find this answer even though we do not have an equation to work with. We can use the fact that $f^{-1}(y) = x$. This means that $f^{-1}(14) = 10$.

EXAMPLE: Verify the following are inverses: $f(x)=\sqrt{x-3}$ and $f^{-1}(x)=x^2+3$

We need to show $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$.

We need to show both sides are true:

$$\begin{aligned} &f(f^{-1}(x)) \\ &= f(x^2+3) \\ &= \sqrt{(x^2+3)-3} \\ &= \sqrt{x^2} = x \end{aligned} \qquad \begin{aligned} &f^{-1}(f(x)) \\ &= f^{-1}(\sqrt{x-3}) \\ &= \left(\sqrt{x-3}\right)^2+3 \\ &= x-3+3 = x \end{aligned}$$

Both sides are equal to x, so we have verified they are inverses.

<u>How to find an inverse algebraically</u>:

Step 1: Replace $f(x)$ with y.
Step 2: Switch x and y.
Step 3: Solve for y.
Step 4: Replace y with $f^{-1}(x)$.

EXAMPLE: Given $f(x)=2x-5$ find $f^{-1}(x)$. Then verify your answer is correct.

We will follow our four steps to find the inverse.

Step 1: $y=2x-5$
Step 2: $x=2y-5$
Step 3: $x+5=2y$

$$\frac{x+5}{2}=y$$

Step 4: $\frac{x+5}{2}=f^{-1}(x)$

Now we need to verify our answer is correct.

We need to show $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$.

We need to show both sides are true:

$$\begin{aligned} &f(f^{-1}(x)) \\ &= f\left(\frac{x+5}{2}\right) \\ &= 2\left(\frac{x+5}{2}\right)-5 \\ &= x+5-5 = x \end{aligned} \qquad \begin{aligned} &f^{-1}(f(x)) \\ &= f^{-1}(2x-5) \\ &= \frac{(2x-5)+5}{2} = \frac{2x}{2} = x \end{aligned}$$

So our answer is correct.

EXAMPLE: Given $f(x)=\sqrt{x+7}$ find $f^{-1}(x)$. Then verify your answer is correct.

Step 1: $y=\sqrt{x+7}$

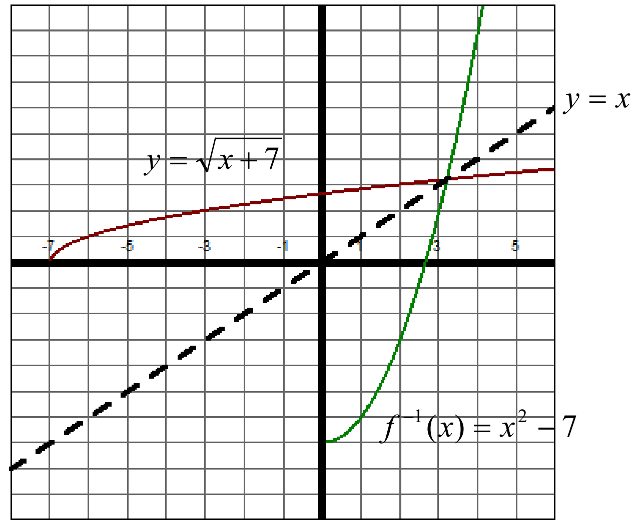
Step 2: $x=\sqrt{y+7}$

Step 3: $(x)^2=\left(\sqrt{y+7}\right)^2$

$x^2=y+7$

$x^2-7=y$

Step 4: $x^2-7=f^{-1}(x),\quad x\ge 0$



Now let's discuss $f(x)=\sqrt{x+7}$. This is a one-to-one function and the graph is the same as the graph of $f(x)=\sqrt{x}$ with a shift of 7 to the left. The domain of $f(x)$ is $(-7,\infty)$ and the range is $(0,\infty)$. When looking at the inverse, the domain and range switch. Therefore the domain of $f^{-1}(x)$ is $(0,\infty)$ and the range is $(-7,\infty)$. So for our final answer, we will write $f^{-1}(x)=x^2-7$, where $x\ge 0$. We also need this restriction because $y=x^2-7$ is not one-to-one without a restricted domain. As you can see in the graph above, with the restriction of $x\ge 0$ the inverse will lay perfectly on top of y when folding the graph along the line $y=x$.
Now we need to verify our answer is correct.

We need to show $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$.

We need to show both sides are true:

$$f(f^{-1}(x)) = f\left(x^2-7\right) = \sqrt{(x^2-7)+7} = \sqrt{x^2}=x$$

$$f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+7}\right) = \left(\sqrt{x+7}\right)^2-7 = x+7-7=x$$

So our answer is correct.

EXAMPLE: Given $f(x)=\dfrac{2x-3}{x+4}$ find $f^{-1}(x)$. You do not need to verify this one.

Step 1: $y=\dfrac{2x-3}{x+4}$

Step 2: $x=\dfrac{2y-3}{y+4}$ Now to solve this one let's first multiply both sides by y + 4 to cancel the fraction.

Step 3: $x(y+4)=2y-3$ Now we need to solve for y. First distribute.

$xy+4x=2y-3$ Let's get all the terms that have y in it to one side of the equation.

$xy-2y=-4x-3$ Now factor out a y.

$y(x-2)=-4x-3$ Now divide both sides by x – 2 and replace y with $f^{-1}(x)$.

Step 4: $f^{-1}(x)=\dfrac{-4x-3}{x-2}$

EXAMPLE: Given $f(x) = \dfrac{3x-5}{2x-3}$ find $f^{-1}(x)$. You do not need to verify this one.

Step 1: $y = \dfrac{3x-5}{2x-3}$

Step 2: $x = \dfrac{3y-5}{2y-3}$ Now to solve this one let's first multiply both sides by 2y – 3 to cancel the fraction.

Step 3: $x(2y-3) = 3y-5$ Now we need to solve for y. First distribute.

$2xy - 3x = 3y - 5$ Let's get all the terms that have y in it to one side of the equation.

$2xy - 3y = 3x - 5$ Now factor out a y.

$y(2x-3) = 3x-5$ Now divide both sides by 2x – 3 and replace y with $f^{-1}(x)$.

Step 4: $f^{-1}(x) = \dfrac{3x-5}{2x-3}$ The $f(x)$ graph was already symmetric to the line $y = x$ so its inverse is itself!