5.3 Exponential Functions

In this section we will be working with exponents, so here is a quick review of the exponent rules:

Laws of Exponents

$a^s \cdot a^t = a^{s+t}$	Example: $2^5 \cdot 2^3 = 2^{5+3} = 2^8$
$\dfrac{a^s}{a^t} = a^{s-t}$	Example: $\dfrac{2^6}{2^3} = 2^{6-3} = 2^3$
$(a^s)^t = a^{s \cdot t}$	Example: $(2^3)^5 = 2^{3 \cdot 5} = 2^{15}$
$(a \cdot b)^s = a^s \cdot b^s$	Example: $(2 \cdot 3)^5 = 2^5 3^5$
$1^s = 1$	Example: $1^{34} = 1$
$a^0 = 1$	Example: $4^0 = 1,\ \pi^0 = 1$
$a^{-s} = \dfrac{1}{a^s}$	Example: $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$
$\left(\dfrac{a}{b}\right)^{-s} = \left(\dfrac{b}{a}\right)^s$	Example: $\left(\dfrac{2}{3}\right)^{-2} = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$

EXAMPLE: Approximate $5^{\sqrt{3}}$ using a calculator. Round your answer to three decimal places.

To enter this on the calculator, first enter 5. Then you want to look for the ^ key. On some calculators you may see a y^x or x^y. All of these mean the same thing. Hit this key and then enter the $\sqrt{3}$. If your calculator displays what you are entering in, then you will be able to hit the square root key first and then the 3. Otherwise you may need to enter a 3 first and then the square root. Once this is done, then press enter again to display the answer. You should get 16.242 rounded to three places.

EXAMPLE: Approximate $e^{1.6}$ using a calculator. Round your answer to two decimal places.

To enter this on the calculator, look for the ln button on your calculator. Right above it you should see either $e^{\wedge}$, e^x, or e^y depending on what calculator you have. You will need to hit the shift or 2nd key to get this on the screen. Enter this and then 1.6. Your answer should be 4.95.

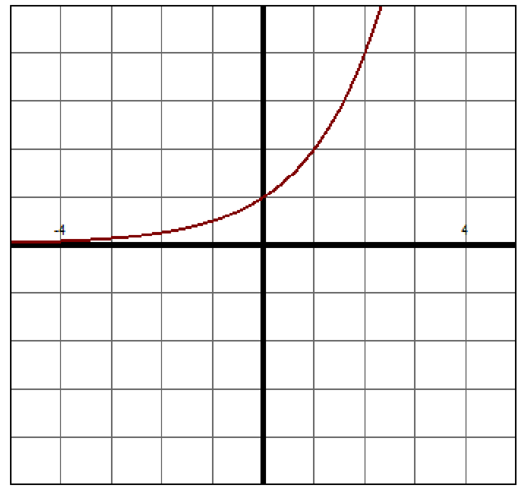
If you get an error on your calculator then you may need to type 1.6 first and then hit the e key.

Exponential function: $y = b^x$

We will look at a specific exponential function to see its characteristics. To do this we will make a table. Then we will plot the points. The graph will be a curved line:

Graph of $y = 2^x$

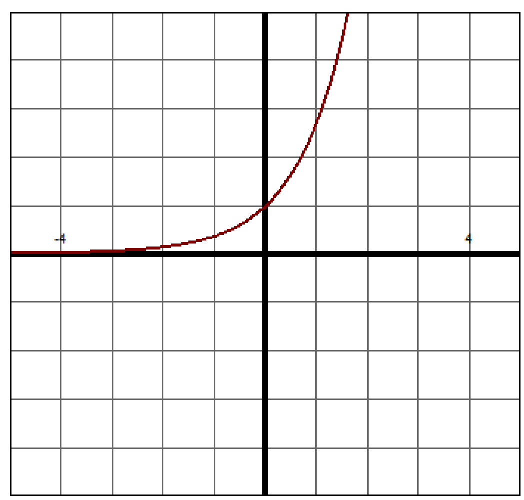
x	$y = 2^x$	(x, y)
-2	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$y = 2^0 = 1$	$(0, 1)$
1	$y = 2^1 = 2$	$(1, 2)$
2	$y = 2^2 = 4$	$(2, 4)$



Notice from the graph of $y = 2^x$ that the y-intercept is (0, 1). This will always be the case for exponential functions. Also notice that there is a horizontal asymptote at y = 0.
Let's now look at the graph of $y = e^x$. To get the letter e on your calculator, look where your LN button is and probably right above it should be a e^ key. Hit this and then the number you want to raise e to. Let's make a table:

Graph of $y = e^x$

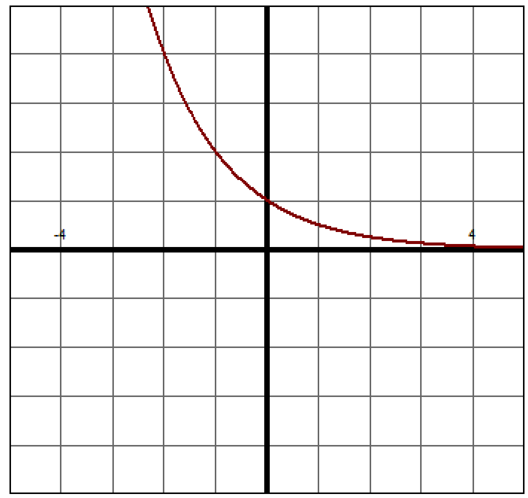
x	$y = e^x$	(x, y)
-2	$y = e^{-2} = \frac{1}{e^2} = \frac{1}{7.39} = 0.135$	$(-2, .0.135)$
-1	$y = e^{-1} = \frac{1}{e^1} = \frac{1}{2.72} = 0.368$	$(-1, 0.368)$
0	$y = e^0 = 1$	$(0, 1)$
1	$y = e^1 = 2.72$	$(0, 2.72)$
2	$y = e^2 = 7.39$	$(2, 7.39)$



This graph still has a y-intercept of (0, 1) and a horizontal asymptote at y = 0. The difference here is that this graph is steeper. So the larger the number is on the bottom, the steeper the graph is. What if we have a fraction instead of a number? Let's find out.

Graph of $y = \left(\frac{1}{2}\right)^x$

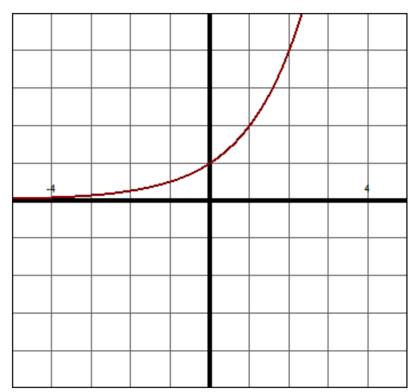
x	$y = \left(\frac{1}{2}\right)^x$	$(x,\ y)$
-2	$y = \left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 4$	$(-2,\ 4)$
-1	$y = \left(\frac{1}{2}\right)^{-1} = \left(\frac{2}{1}\right)^1 = 2$	$(-1,\ 2)$
0	$y = \left(\frac{1}{2}\right)^0 = 1$	$(0,\ 1)$
1	$y = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1,\ \frac{1}{2}\right)$
2	$y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2,\ \frac{1}{4}\right)$



Notice that a fraction causes the graph to flip about the vertical axis. This is because $y = \left(\frac{1}{2}\right)^x = \left(\frac{2}{1}\right)^{-x} = 2^{-x}$.

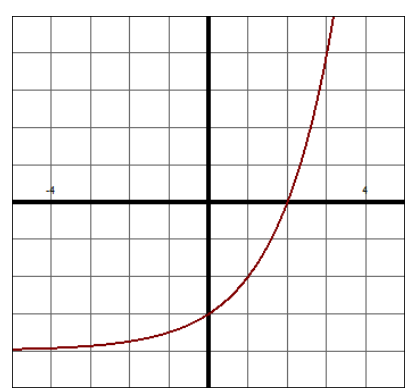
Since the exponent is negative this means we flip the graph of $y = 2^x$ about the vertical axis.

EXAMPLE: Graph using transformations: $y = 2^x - 4$
Indicate the domain and range. State the horizontal asymptote.

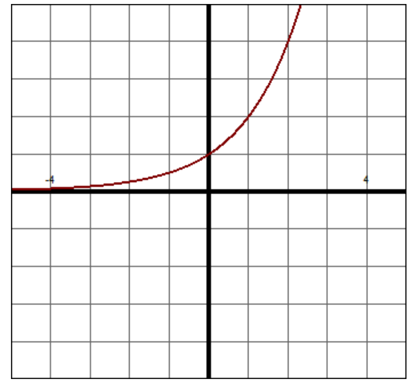


We start with our base graph of $y = 2^x$. The –4 is a transformation that will move the entire graph down 4 places. Therefore the original y-intercept at (0, 1) gets moved down 4 places to (0, –3). The horizontal asymptote also move down 4 units. In the base graph of $y = 2^x$ the horizontal asymptote was at $y = 0$, so after the transformation the horizontal asymptote is $y = -4$.

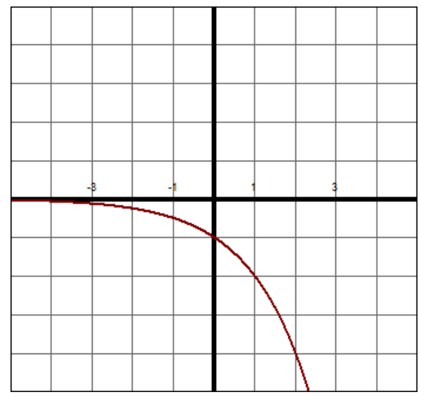
There are no restrictions on which x-values can be used, so the domain is $(-\infty, \infty)$. The range is the y-values the graph uses. After drawing our graph we see the range is $(-4, \infty)$



EXAMPLE: Graph using transformations: $y = -2^x$.
Indicate the domain and range. State the horizontal asymptote.



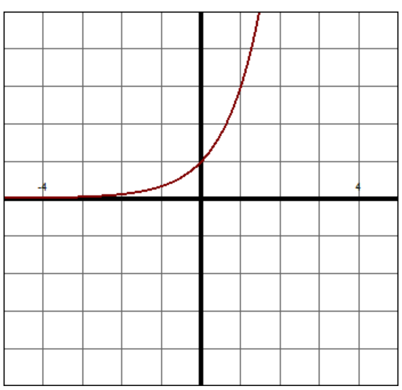
We start with the base graph $y = 2^x$. The negative this time is in front of the number being raised to the power of x. This is another transformation. When this happens it will flip the graph over the horizontal axis. Notice that the original y-intercept is also reflected over the horizontal axis. It was originally at (0, 1) but now it is at (0, –1).



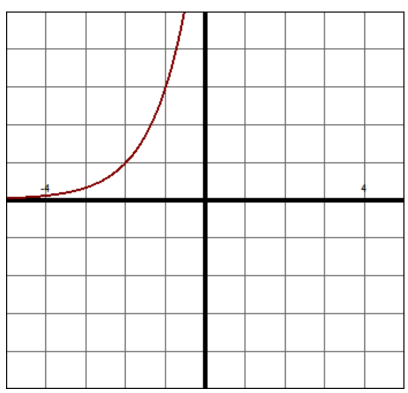
The horizontal asymptote did not shift, so it is the same as our base graph, which is y = 0.

No restrictions on x-values, so the domain is $(-\infty, \infty)$.
We see that the graph is only using negative values, so the range is $(-\infty, 0)$.

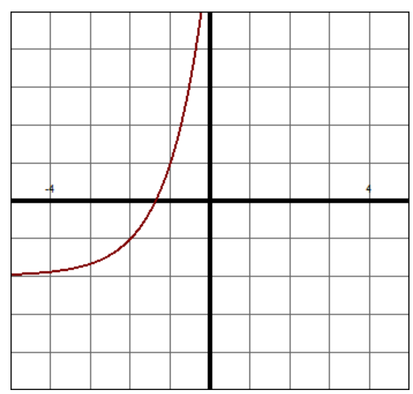
EXAMPLE: Graph using transformations: $y = 3^{x+2} - 2$.
Indicate the domain and range. State the horizontal asymptote.



Start with our base graph $y = 3^x$
Our y-intercept is at (0, 1).



Next we graph $y = 3^{x+2}$ which moves the graph $y = 3^x$ two places to the left. Notice the point (0, 1) got shifted 2 places to the left, so the new point is now at (–2, 1).



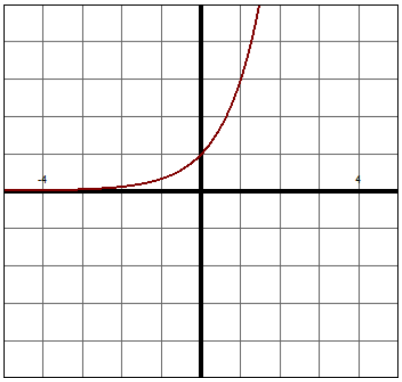
Now we shift the graph $y = 3^{x+2}$ down 2 units. So the point at (–2, 1) is now at (–2, –1).

In our final answer, the horizontal asymptote is y = –2. The domain is $(-\infty, \infty)$ and range $(-2, \infty)$

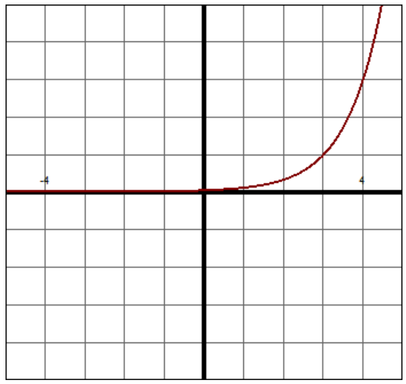
EXAMPLE: Graph using transformations: $y = \left(\frac{1}{3}\right)^{x-3} + 1$.

Indicate the domain and range. State the horizontal asymptote.

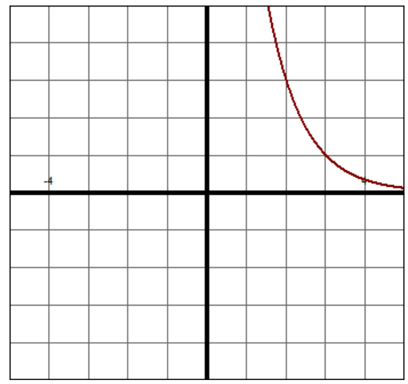
First we want to rewrite the original as $y = \left(3^{-1}\right)^{x-3} + 1$. Then we write $y = 3^{-(x-3)} + 1$



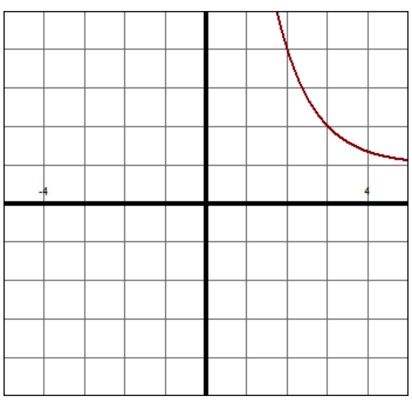
Start with our base graph $y = 3^x$
Our y-intercept is at (0, 1).



Next we graph $y = 3^{x-3}$ which moves the graph $y = 3^x$ three places to the right. Notice the point (0, 1) got shifted 3 places to the right, so the new point is now at (3, 1).



To graph $y = 3^{-(x-3)}$ we will flip $y = 3^{x-3}$ over the vertical axis. Notice our point at (3, 1) is still at the same location.



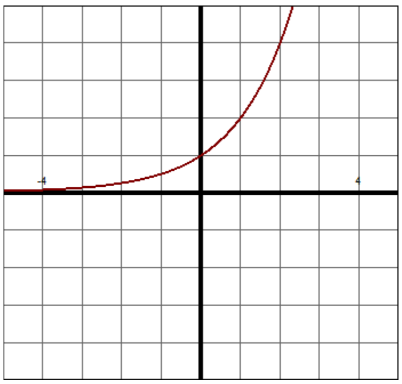
Now we move the graph of $y = 3^{-(x-3)}$ up one unit. The point (3, 1) has move up one unit to (3, 2).
This graph is our final answer.

In our final answer, the horizontal asymptote is y = 1. The domain is $(-\infty, \infty)$ and range $(1, \infty)$

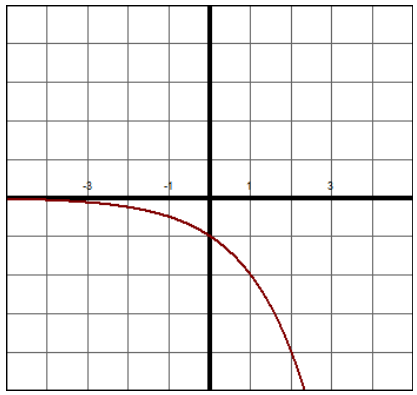
EXAMPLE: Graph using transformations: $y = -\left(\frac{1}{2}\right)^x + 3$.

Indicate the domain and range. State the horizontal asymptote.

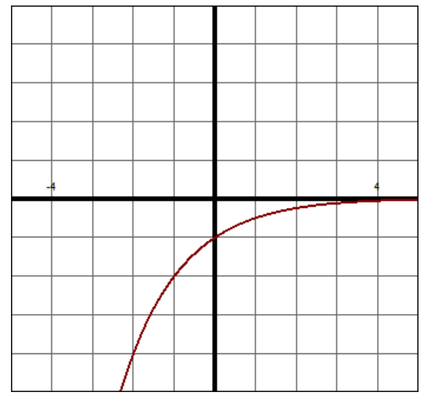
First we want to rewrite the original as $y = -\left(2^{-1}\right)^x + 3$. Then we write $y = -2^{-x} + 3$



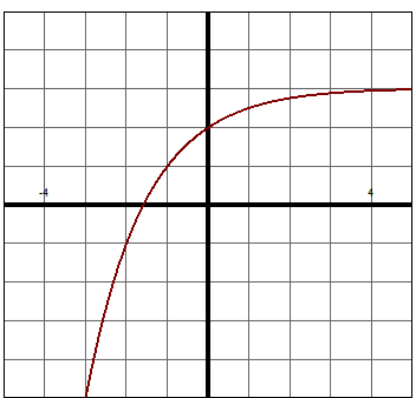
Start with our base graph $y = 2^x$
Our y-intercept is at (0, 1).



Next we graph $y = -2^x$ which flips the graph $y = 2^x$ over the horizontal axis. Notice the point (0, 1) got flipped over the horizontal axis, so the new point is now at (0, –1).



To graph $y = -2^{-x}$ we will flip $y = -2^x$ over the vertical axis. Notice our point at (0, –1) is still at the same location.



Now we move the graph of $y = -2^{-x}$ up three units. The point (0, –1) has move up three units to (0, 2).
This graph is our final answer.

In our final answer, the horizontal asymptote is y = 3. The domain is $(-\infty, \infty)$ and range $(-\infty, 3)$.

Equal Bases Property (The Equivalence Property of Exponential Expressions)

If $a^u = a^v$ then u = v.

EXAMPLE: Solve: $4^{x-2} = 64$.

In order to solve this, we must make both the bases the same. Since there is a 4 on the left hand side, I want to write 64 as 4 raised to some power. It is known that $4^3 = 64$ so we can now rewrite our equation:

$4^{x-2} = 4^3$. The Equivalence Property of Exponential Expressions states that if the bases are the same then we can set the exponents equal to each other. If we do this we will have x – 2 = 3. Solving this we get x = 5.

EXAMPLE: Solve: $\left(\frac{1}{2}\right)^{1-x} = 4$.

We need to get both the bases to be the same. Both sides can be written with a base of 2. If we flip the fraction of ½ over we will change the sign of the exponent. We can also write 4 as 2^2. Our equation is now:
$\left(\frac{2}{1}\right)^{-(1-x)} = 2^2$. Now both bases are 2. We can set the exponents equal to each other. You will get –(1 – x) = 2. Solving this you will get x = 3.

EXAMPLE: Solve: $27^{x+2} = 9^{2x}$.

Again we need the same base. 27 can't be written as 9 to a power, so we need something smaller. Each of these can be written with a base of 3:

$(3^3)^{x+2} = (3^2)^{2x}$ We are raising a power to another power and when you do, multiply the exponents.

$3^{3x+6} = 3^{4x}$ Now the bases are the same so let's set the exponents equal: 3x + 6 = 4x. Solving for x we get x = 6.

EXAMPLE: Solve: $3^{4y-1} = \sqrt{3}$.

We can change the right hand side to be $3^{\frac{1}{2}}$. Then our equation becomes $3^{4y-1} = 3^{\frac{1}{2}}$. The bases are the same, so we can set the exponents together. You will get $4y - 1 = \frac{1}{2}$. To solve this, multiply both sides by 2 to clear the fractions. You will get $8y - 2 = 1$. Then solve for y. The answer is $y = \frac{3}{8}$.

EXAMPLE: Solve: $e^{3x+5} = 1$.

This may seem as if we cannot solve this because the bases are not the same, however there is a way we can write these with the same bases. Recall that anything raised to a power of 0 is 1, so we rewrite the right side of the equation to be this: $e^{3x+5} = e^0$. Now they have equal bases, so we can set the exponents equal to each other. You will get $3x + 5 = 0$. Solving, we get $x = -\frac{5}{3}$.

EXAMPLE: Solve: $\left(2^x\right)^{x-2} = 8$.

Since we are raising a power to another power we need to multiply the exponents. I also want to change the 8 into 2^3 so we can have the same base on both sides. Now our equation becomes $2^{x^2-2x} = 2^3$. Since the bases are the same we can now set the exponents equal to each other. The equation becomes $x^2 - 2x = 3$. In order to solve this you must set it equal to zero and solve. We have $x^2 - 2x - 3 = 0$. Factoring we get $(x+1)(x-3) = 0$ and so x = -1 and x = 3.