5.3 Exponential Functions

In this section we will be working with exponents, so here is a quick review of the exponent rules:

Laws of Exponents

$a^s \cdot a^t = a^{s+t}$ Example: $2^5 \cdot 2^3 = 2^{5+3} = 2^8$

$\dfrac{a^s}{a^t} = a^{s-t}$ Example: $\dfrac{2^6}{2^3} = 2^{6-3} = 2^3$

$(a^s)^t = a^{s \cdot t}$ Example: $(2^3)^5 = 2^{3 \cdot 5} = 2^{15}$

$(a \cdot b)^s = a^s \cdot b^s$ Example: $(2 \cdot 3)^5 = 2^5 3^5$

$1^s = 1$ Example: $1^{34} = 1$

$a^0 = 1$ Example: $4^0 = 1$, $\pi^0 = 1$

$a^{-s} = \dfrac{1}{a^s}$ Example: $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

$\left(\dfrac{a}{b}\right)^{-s} = \left(\dfrac{b}{a}\right)^s$ Example: $\left(\dfrac{2}{3}\right)^{-2} = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$

EXAMPLE: Approximate $5^{\sqrt{3}}$ using a calculator. Round your answer to three decimal places.

To enter this on the calculator, first enter 5. Then you want to look for the ^ key. On some calculators you may see a y^x or x^y. All of these mean the same thing. Hit this key and then enter the $\sqrt{3}$. If your calculator displays what you are entering in, then you will be able to hit the square root key first and then the 3. Otherwise you may need to enter a 3 first and then the square root. Once this is done, then press enter again to display the answer. You should get 16.242 rounded to three places.

EXAMPLE: Approximate $e^{1.6}$ using a calculator. Round your answer to two decimal places.

To enter this on the calculator, look for the ln button on your calculator. Right above it you should see either $e^{\wedge}$, e^x, or e^y depending on what calculator you have. You will need to hit the shift or 2^{nd} key to get this on the screen. Enter this and then 1.6. Your answer should be 4.95.

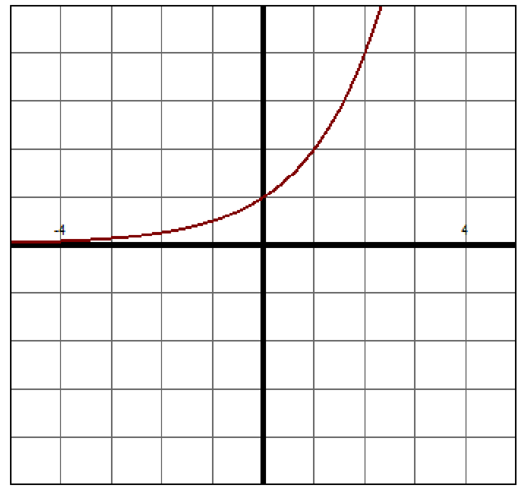
If you get an error on your calculator then you may need to type 1.6 first and then hit the e key.

Exponential function: $y = b^x$

We will look at a specific exponential function to see its characteristics. To do this we will make a table. Then we will plot the points. The graph will be a curved line:

Graph of $y = 2^x$

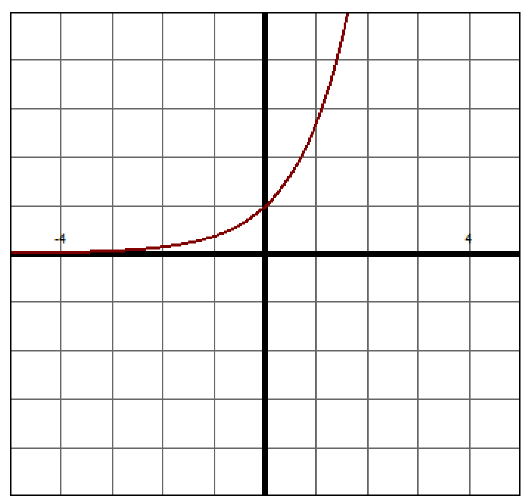
x	$y = 2^x$	(x, y)
-2	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$y = 2^0 = 1$	$(0, 1)$
1	$y = 2^1 = 2$	$(1, 2)$
2	$y = 2^2 = 4$	$(2, 4)$



Notice from the graph of $y = 2^x$ that the y-intercept is (0, 1). This will always be the case for exponential functions. Also notice that there is a horizontal asymptote at y = 0.
Let's now look at the graph of $y = e^x$. To get the letter e on your calculator, look where your LN button is and probably right above it should be a e^ key. Hit this and then the number you want to raise e to. Let's make a table:

Graph of $y = e^x$

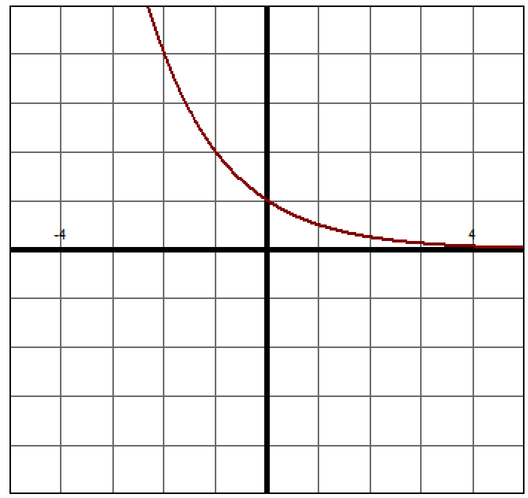
x	$y = e^x$	(x, y)
-2	$y = e^{-2} = \frac{1}{e^2} = \frac{1}{7.39} = 0.135$	$(-2, .0.135)$
-1	$y = e^{-1} = \frac{1}{e^1} = \frac{1}{2.72} = 0.368$	$(-1, 0.368)$
0	$y = e^0 = 1$	$(0, 1)$
1	$y = e^1 = 2.72$	$(0, 2.72)$
2	$y = e^2 = 7.39$	$(2, 7.39)$



This graph still has a y-intercept of (0, 1) and a horizontal asymptote at y = 0. The difference here is that this graph is steeper. So the larger the number is on the bottom, the steeper the graph is. What if we have a fraction instead of a number? Let's find out.

Graph of $y = \left(\frac{1}{2}\right)^x$

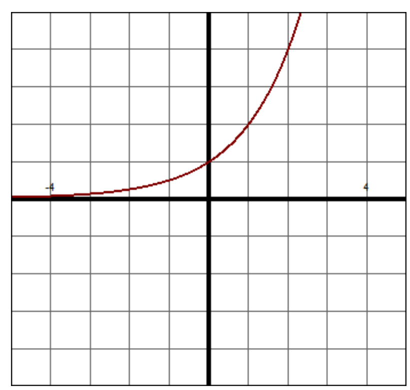
x	$y = \left(\frac{1}{2}\right)^x$	$(x,\ y)$
-2	$y = \left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^{2} = 4$	$(-2,\ 4)$
-1	$y = \left(\frac{1}{2}\right)^{-1} = \left(\frac{2}{1}\right)^{1} = 2$	$(-1,\ 2)$
0	$y = \left(\frac{1}{2}\right)^{0} = 1$	$(0,\ 1)$
1	$y = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$	$\left(1,\ \frac{1}{2}\right)$
2	$y = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$	$\left(2,\ \frac{1}{4}\right)$



Notice that a fraction causes the graph to flip about the vertical axis. This is because $y = \left(\frac{1}{2}\right)^x = \left(\frac{2}{1}\right)^{-x} = 2^{-x}$.

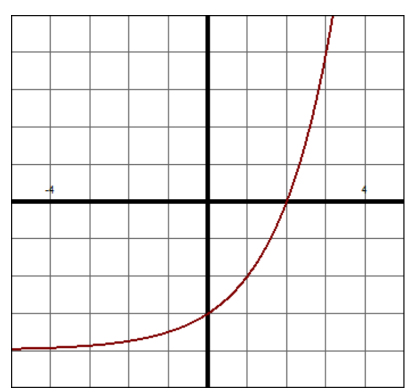
Since the exponent is negative this means we flip the graph of $y = 2^x$ about the vertical axis.

EXAMPLE: Graph using transformations: $y = 2^x - 4$
Indicate the domain and range. State the horizontal asymptote.



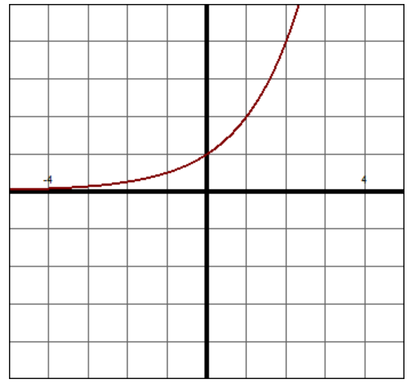
We start with our base graph of $y = 2^x$. The –4 is a transformation that will move the entire graph down 4 places. Therefore the original y-intercept at (0, 1) gets moved down 4 places to (0, –3). The horizontal asymptote also move down 4 units. In the base graph of $y = 2^x$ the horizontal asymptote was at $y = 0$, so after the transformation the horizontal asymptote is $y = -4$.

There are no restrictions on which x-values can be used, so the domain is $(-\infty, \infty)$. The range is the y-values the graph uses. After drawing our graph we see the range is $(-4, \infty)$



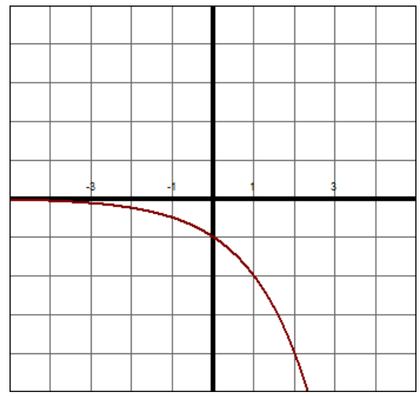
EXAMPLE: Graph using transformations: $y = -2^x$.
Indicate the domain and range. State the horizontal asymptote.

We start with the base graph $y = 2^x$. The negative this time is in front of the number being raised to the power of x. This is another transformation. When this happens it will flip the graph over the horizontal axis. Notice that the original y-intercept is also reflected over the horizontal axis. It was originally at (0, 1) but now it is at (0, –1).

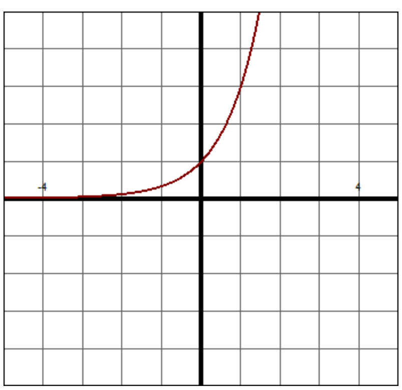


The horizontal asymptote did not shift, so it is the same as our base graph, which is y = 0.

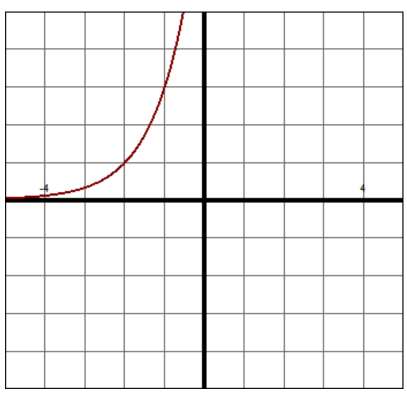
No restrictions on x-values, so the domain is $(-\infty, \infty)$.
We see that the graph is only using negative values, so the range is $(-\infty, 0)$.



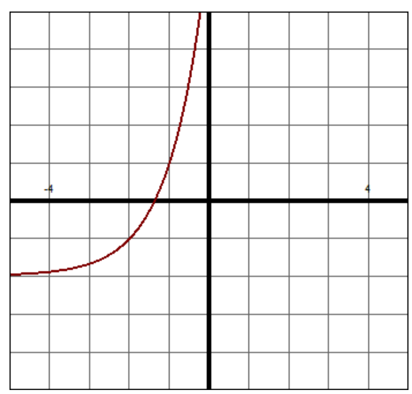
EXAMPLE: Graph using transformations: $y = 3^{x+2} - 2$.
Indicate the domain and range. State the horizontal asymptote.



Start with our base graph $y = 3^x$
Our y-intercept is at (0, 1).



Next we graph $y = 3^{x+2}$ which moves the graph $y = 3^x$ two places to the left. Notice the point (0, 1) got shifted 2 places to the left, so the new point is now at (–2, 1).



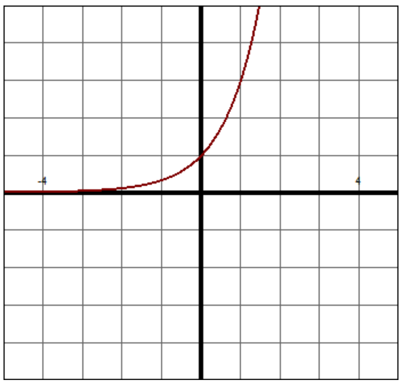
Now we shift the graph $y = 3^{x+2}$ down 2 units. So the point at (–2, 1) is now at (–2, –1).

In our final answer, the horizontal asymptote is y = –2. The domain is $(-\infty, \infty)$ and range $(-2, \infty)$

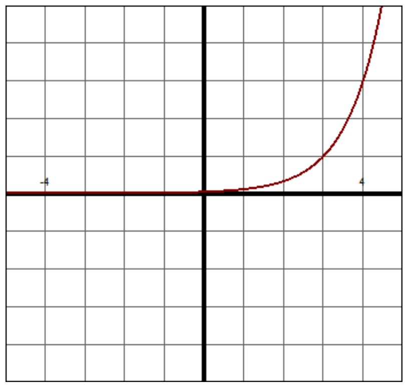
EXAMPLE: Graph using transformations: $y = \left(\frac{1}{3}\right)^{x-3} + 1$.

Indicate the domain and range. State the horizontal asymptote.

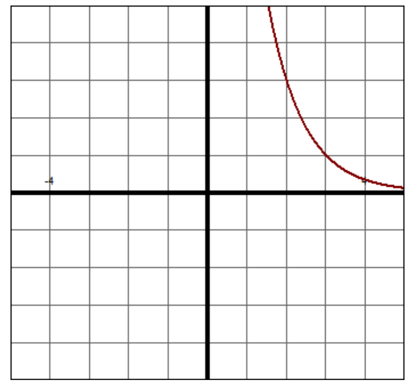
First we want to rewrite the original as $y = \left(3^{-1}\right)^{x-3} + 1$. Then we write $y = 3^{-(x-3)} + 1$



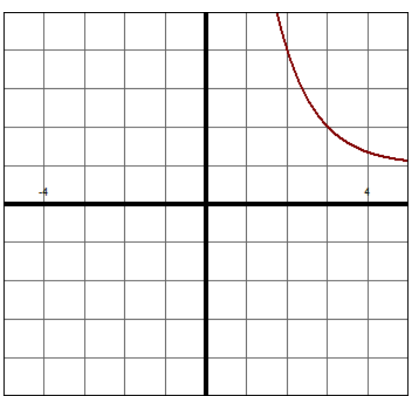
Start with our base graph $y = 3^x$
Our y-intercept is at (0, 1).



Next we graph $y = 3^{x-3}$ which moves the graph $y = 3^x$ three places to the right. Notice the point (0, 1) got shifted 3 places to the right, so the new point is now at (3, 1).



To graph $y = 3^{-(x-3)}$ we will flip $y = 3^{x-3}$ over the vertical axis. Notice our point at (3, 1) is still at the same location.



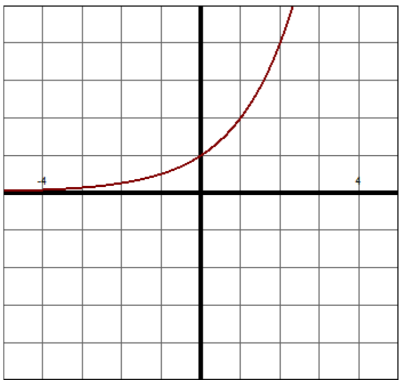
Now we move the graph of $y = 3^{-(x-3)}$ up one unit. The point (3, 1) has move up one unit to (3, 2). This graph is our final answer.

In our final answer, the horizontal asymptote is y = 1. The domain is $(-\infty, \infty)$ and range $(1, \infty)$

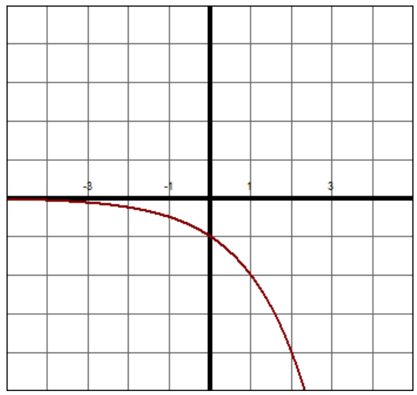
EXAMPLE: Graph using transformations: $y = -\left(\frac{1}{2}\right)^x + 3$.

Indicate the domain and range. State the horizontal asymptote.

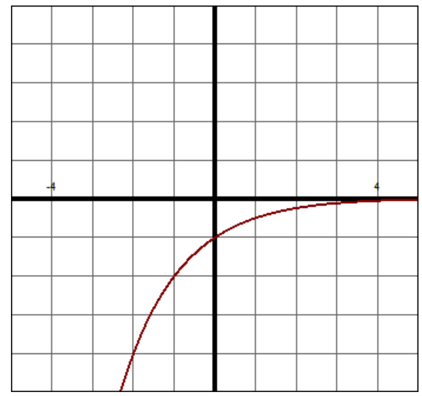
First we want to rewrite the original as $y = -\left(2^{-1}\right)^x + 3$. Then we write $y = -2^{-x} + 3$



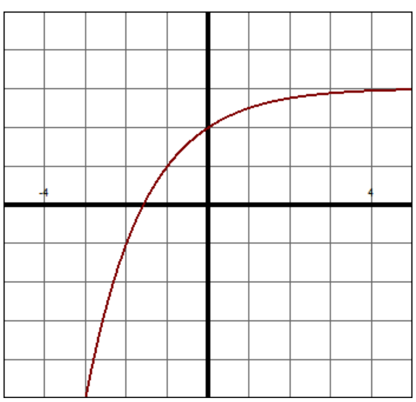
Start with our base graph $y = 2^x$
Our y-intercept is at (0, 1).



Next we graph $y = -2^x$ which flips the graph $y = 2^x$ over the horizontal axis. Notice the point (0, 1) got flipped over the horizontal axis, so the new point is now at (0, –1).



To graph $y = -2^{-x}$ we will flip $y = -2^x$ over the vertical axis. Notice our point at (0, –1) is still at the same location.



Now we move the graph of $y = -2^{-x}$ up three units. The point (0, –1) has move up three units to (0, 2). This graph is our final answer.

In our final answer, the horizontal asymptote is y = 3. The domain is $(-\infty, \infty)$ and range $(-\infty, 3)$.

Compound Interest

The amount A after t years due to a principal P invested at an annual interest rate r (expressed as a decimal) compounded n times per year is $A = P \cdot \left(1+\frac{r}{n}\right)^{nt}$. The compound interest formula actually starts with the simple interest formula, $I = Prt$. Interest is calculated once, and then this interest is added to the principal, and the formula is repeated again. This process continues until it is compounded n times. So instead of that long process, we have the compound interest formula.

In working with these kinds of problems, they will give you different payment periods as listed below, which also tell you what to enter for n:

Annually: Once a year, $n = 1$
Semiannually: Twice a year, $n = 2$
Quarterly: Four times a year, $n = 4$
Monthly: 12 times a year, $n = 12$
Weekly: 52 times a year, $n = 52$
Daily: 365 days per year, $n = 365$

EXAMPLE: $300 is invested at 12% compounded monthly for $1\frac{1}{2}$ years. Find the amount that results from this investment.

Here $n = 12$ since it is compounded monthly. The principal is P = $300, and $r = 0.12$ and $t = 1.5$. We will put these into the compound interest formula: $A = 300 \cdot \left(1+\frac{0.12}{12}\right)^{12\cdot(1.5)}$. First simplify inside the parenthesis and also multiply the n and t together in the exponent position: $A = 300 \cdot (1.01)^{18}$. Now we will raise 1.01 to the power of 18. You will get $A = 300 \cdot 1.196147...$ Now multiply to get our answer: A = $358.84.

Continuous Compounding

If you take the same compound interest formula as above and have n go to infinity, you will get $A = P \cdot e^{rt}$, which is the formula for compounding continuously.

EXAMPLE: $300 is invested at 12% compounded continuously for $1\frac{1}{2}$ years. Find the amount that results from this investment.

Since we are compounding continuously, we will use $A = P \cdot e^{rt}$, which P = $300, $r = 0.12$, and $t = 1.5$. Plug it in to get: $A = 300 \cdot e^{0.12\cdot(1.5)}$. Work the exponent part first: $A = 300 \cdot e^{0.18}$. Now we will raise e to the power of 0.18. Use the e^ key on your calculator: $A = 300 \cdot (1.197217...)$. So A = $359.17. Notice that this yields a slightly higher investment when compared to compounding monthly.

EXAMPLE: Determine the rate that represents the better deal:

9% compounded quarterly or $9\frac{1}{4}\%$ compounded annually

For this problem, they do not give you a principal or time. To make it easy, let's assume $P = 1000$ and $t = 1$. Then we will apply the compound interest formula for each one separately. Note that we can use any value for P and t as long as the same ones are used for each rate. Now let's calculate:

9% compounded quarterly ($n = 4$, $r = 0.09$) $\quad A = 1000 \cdot \left(1 + \frac{0.09}{4}\right)^{4(1)} = 1000(1.0225)^4 = \$1093.08$

$9\frac{1}{4}\%$ compounded annually ($n = 1$, $r = 0.0925$) $\quad A = 1000 \cdot \left(1 + \frac{0.0925}{1}\right)^{1(1)} = 1000(1.0925) = \$1092.50$

The first option (9% compounded quarterly) is the better deal. It may not seem like much savings with such a small principal, however if you have a principal of, say, a million dollars, then the savings become more significant.

EXAMPLE: What will a \$90,000 house cost 10 years from now if the price appreciation for homes over that period averages 4% compounded annually?

We will use the compound interest formula with $P = 90000$, $t = 10$, $n = 1$, and $r = 0.04$:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A = 90000\left(1 + \frac{0.04}{1}\right)^{1 \cdot 10}$$
$$A = 90000(1 + 0.04)^{10}$$
$$A = 90000(1.04)^{10}$$
$$A \approx \$133,221.99$$