5.4 Logarithmic Functions

In a previous section we looked at inverses. In order to find a inverse we need to switch *x* and *y*.

Suppose we wanted to find the inverse of our exponent function, $y = b^x$. First we need to switch *x* and *y*. We will get $x = b^y$. How do we solve for *y*? This is where we need logarithms, which are a way to solve for an exponent.

With logarithms there are two forms: exponential and logarithmic.

Exponential form: $x = b^y$

Logarithmic form: $y = \log_b x$

Both of these forms are exactly the same. Let's practice changing between the two forms.

EXAMPLE: Change $7 = \log_m 5$ into exponential form.

Here y = 7, b = m and x = 5. If we put these into the exponential form we get $m^7 = 5$.

EXAMPLE: Change $\log_c 6 = 8$ into exponential form.

Here b = c, y = 8 and x = 6. If we put these into the exponential form we get $c^8 = 6$.

EXAMPLE: Change $q = 1.4^5$ into logarithmic form.

Here x = q, b = 1.4 and y = 5. If we put these into the logarithmic form we get $5 = \log_{1.4} q$.

EXAMPLE: Change $2^d = 8$ into logarithmic form.

Here x = 8, b = 2 and y = d. If we put these into the logarithmic form we get $d = \log_2 8$.

EXAMPLE: Change $x = e^y$ into logarithmic form.

Changing this into logarithmic form we get $y = \log_e x$. This has a special form: $y = \ln x$. This called the **natural logarithm**. This is the same as a logarithm with a base of e.

We can do logs on our calculator. We have a ln key and also a log key. Our calculator will only do base 10 and base e logarithms. Let's practice using the calculator:

EXAMPLE: Find the value of $\log 5$ on a calculator and round to the nearest thousandth.

In your text you will sometimes see logs without bases. If there is no base then this automatically means it is base 10. So we need to find the numerical value of $\log_{10} 5$ on our calculator. If you have a scientific calculator that does not display what you are typing in, you will need to enter the 5 and then hit the log key. For all other calculators, enter log and then 5. You should get 0.699.

EXAMPLE: Find the value of $\ln 7$ on a calculator and round to the nearest hundredth.

Enter 7 and the ln key if you have a scientific calculator the does not display what you are typing in. Otherwise, first enter ln and then 7. You should get 1.95.

EXAMPLE: Find the exact value of $\log_4 64$.

This one we can't do on our calculator like the ones above since we don't have a base 4. What we can do is change it into exponential form and solve. First we will let $y = \log_4 64$. Changing into exponential form we will get $4^y = 64$. We want to use the Equal Bases Property. To do this, we need to rewrite 64 so that it has a base of 4. This is 4^3, since (4)(4)(4) = 64. So since $4^y = 4^3$ we can set the exponents equal, resulting in $y = 3$. So now we know that $\log_4 64 = 3$.

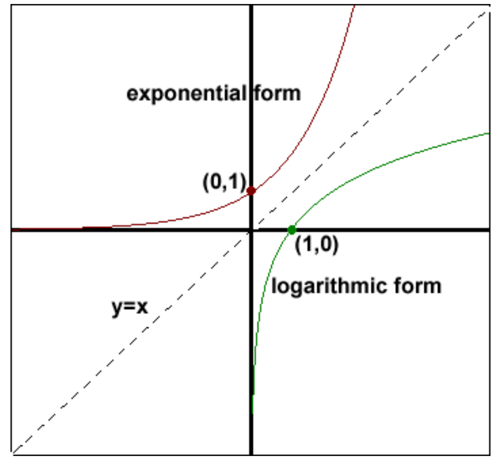
EXAMPLE: Find the exact value of $\log \frac{1}{10000}$.

Remember that this is the same as $.\log_{10} \frac{1}{10000}$. Now we will change it into exponential form and solve. First we will let $y = \log_{10} \frac{1}{10000}$. Changing into exponential form we will get $10^y = \frac{1}{10000}$. We want to use the Equal Bases Property. To do this, we need to rewrite the right hand of the equation so that it has a base of 10. So $10000 = 10^4$. But since it is on the bottom of a fraction then $\frac{1}{10000} = 10^{-4}$. So since $10^y = 10^{-4}$ we can set the exponents equal, resulting in $y = -4$. So now we know that $\log \frac{1}{10000} = -4$.

EXAMPLE: Find the value of $\log 0$ on a calculator and round to the nearest hundredth.

Remember that this is the same as $.\log_{10} 0$. Now we will change it into exponential form and solve. First we will let $y = \log_{10} 0$. Changing into exponential form we will get $10^y = 0$. No number we pick can satisfy this equation, so it is no solution. Verifying on a calculator also gives us the same result. Therefore this tells us that there must be a restriction on what numbers we can use. Let's take a look at the graph to see visually which numbers we are allowed (domain).

Let's try and draw a graph of $y = \log_b x$. In order to do this, we need to first draw the graph of $x = b^y$. Then we need to draw its inverse. We know that the log function is the inverse of the exponential function. We also know that all inverse are reflective about the line y = x. In the graph to the right you can see both graphs drawn. We see that the point (0, 1) and (1, 0) are flipped, which it should since they are inverses. So the graph of $y = \log_b x$ will always cross over the x-axis one unit away from the vertical asymptote. Let's look at the graph of the logarithmic form. The y-axis is a vertical asymptote. There is no horizontal asymptote on the logarithmic graph. From the graph we can conclude:



Domain of $y = \log_b x$ **is x > 0**. Notice that zero is not included since this is a vertical asymptote. Also notice that it doesn't matter what *b* is.

Let's look at some problems that ask you to find the domain.

EXAMPLE: Find the domain of $y = \log_3(6 - x)$ and write your answer in interval notation.

Whatever is inside the parenthesis must be set to be greater than zero since this is the domain: 6 – x > 0. If we subtract 6 from both sides we get –x > –6. Dividing both sides by -1 we get x < 6. Notice we needed to switch the sign because we divided by a negative. We write our answer as $(-\infty,\ 6)$.

EXAMPLE: Find the domain of $y = \ln(2x - 7)$ and write your answer in interval notation.

Even though we have a natural log we will still solve this the same way since a natural log is a just a log with a base of e: 2x – 7 > 0. Solving this we will get $x > \dfrac{7}{2}$. We write our answer as $\left(\dfrac{7}{2},\ \infty\right)$.

EXAMPLE: Find the domain of $y = \log_2\left(\dfrac{x+2}{2-x}\right)$ and write your answer in interval notation.

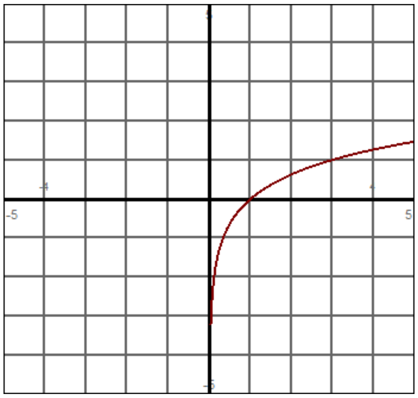
We will have $\dfrac{x+2}{2-x} > 0$. Recall that if we have a fraction with an inequality, then this requires us to set up a table with the plus and minus values. Our critical points will be -2 and 2.

$x+2$	–	+	+
$2-x$	+	+	–

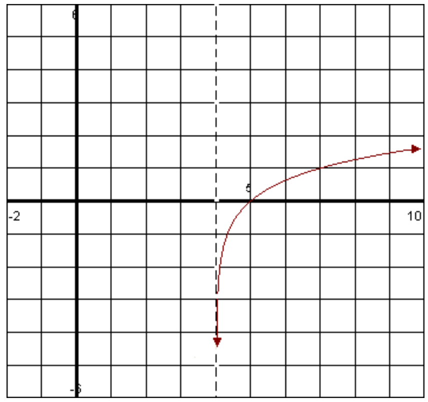
–   -2   +   2   –

So we see that our answer is (-2, 2).

EXAMPLE: Graph using transformations: $y = \log_3(x-4)$ and identify the x-intercept.



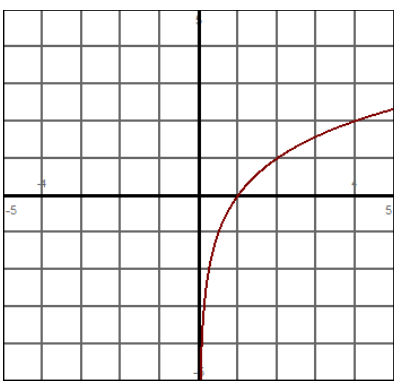
Start with our base graph
$y = \log_b x$



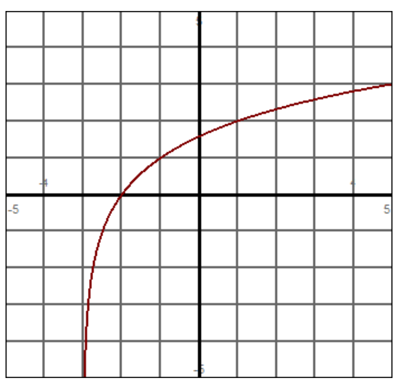
Because of transformation rules, we move $y = \log_b x$ four places to the right. Note the dotted line indicating our vertical asymptote.

This also asks us to find the x-intercept. Notice in our base graph that the graph crosses the *x*-axis one unit to the right of the vertical asymptote. This will also occur in our final graph. Since the vertical asymptote is at $x = 4$, the graph will cross one unit to the right, which will occur at $x = 5$.

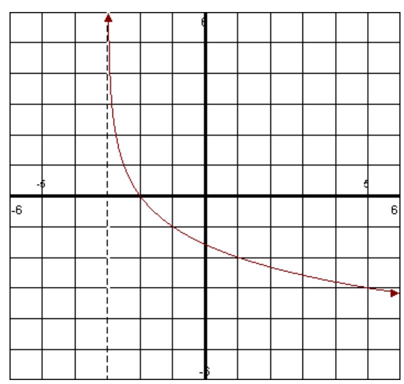
EXAMPLE: Graph using transformations: $y = -\log_2(x+3)$ and identify the x-intercept.



Start with our base graph
$y = \log_b x$



We move $y = \log_b x$ three places to the left. This is the graph of $y = \log_2(x+3)$.

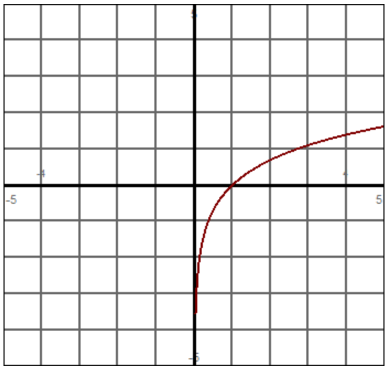


Because of the negative in front of the log, we flip the graph over the horizontal axis. We indicate the vertical asymptote with a dotted line.

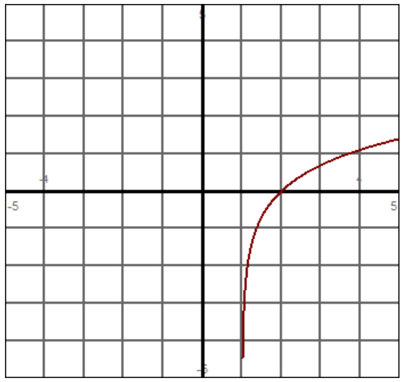
This also asks us to find the x-intercept. Notice in our base graph that the graph crosses the *x*-axis one unit to the right of the vertical asymptote. This will also occur in our final graph. Since the vertical asymptote is at $x = -3$, the graph will cross one unit to the right, which will occur at $x = -2$.

EXAMPLE: Graph using transformations: $y = \ln(1 - x)$ and identify the x-intercept.

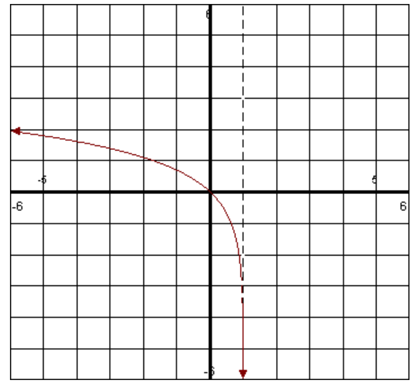
This is equivalent to $y = \log_e(1 - x)$ which will have the same initial shape as the previous two examples. Before we can use transformations we first need to get the x to come first. We will get $y = \log_e(-x + 1)$. Now we factor out a -1: $y = \log_e\left(-(x - 1)\right)$. So we will now graph $y = \ln\left(-(x - 1)\right)$ using transformations.



Start with our base graph $y = \ln x$



We move $y = \ln x$ one place to the right. This is the graph of $y = \ln(x - 1)$.

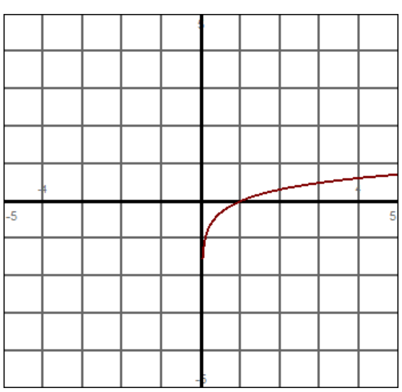


Because of the negative inside of the log, we flip the graph over the vertical axis. We indicate the vertical asymptote with a dotted line.

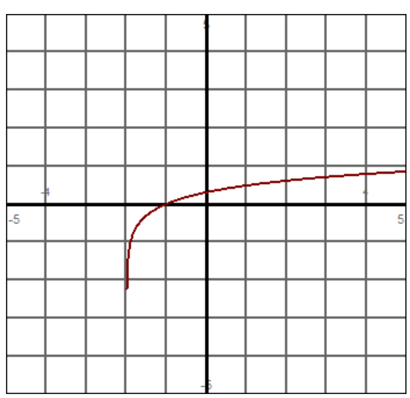
This also asks us to find the x-intercept. Notice that in our final answer the graph crosses the x-axis one unit to the left of the vertical asymptote instead of the right. This is because of the negative inside. Since the vertical asymptote is at $x = 1$, the graph will cross one unit to the right, which will occur at $x = 0$.

EXAMPLE: Graph using transformations: $y = -\log\left(-\left(x + 2\right)\right)$ and identify the x-intercept.

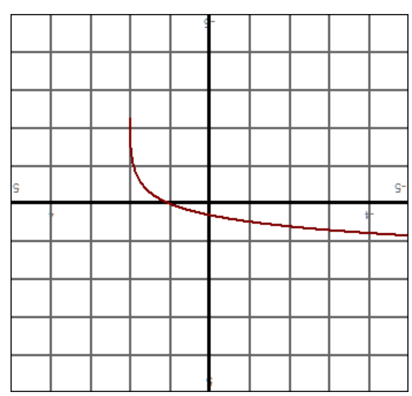
This is written in the correct form, so we can immediately start graphing using transformations. Note that this log has a base of 10.



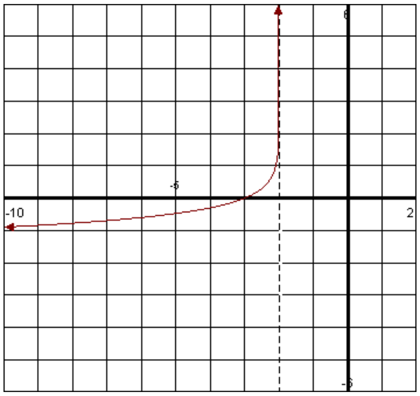
Start with our base graph $y = \log x$



We move $y = \log x$ two places to the left. This is the graph of $y = \log(x + 2)$



Because of the negative outside of the log, we flip the graph over the horizontal axis. This is the graph $y = -\log\left(x + 2\right)$



Because of the negative inside of the log, we flip the graph $y = -\log(x+2)$ over the vertical axis. This is the graph of $y = -\log(-(x+2))$

Notice that in our final answer the graph crosses the x-axis one unit to the left of the vertical asymptote instead of the right. This is because of the negative inside. Since the vertical asymptote is at $x = -2$, the graph will cross one unit to the right, which will occur at $x = -3$.