R.2 Complex Numbers

In this section we will review complex (or imaginary) numbers. First we will start with the basic definition of an imaginary number. We will let $i = \sqrt{-1}$. Normally we would not be able to get a value of this, but in this section we will be working with i and how to simplify expressions involving these kind of numbers. Let's look at what happens when we see different powers of i:

$i^0 = 1$	Anything to the power of 0 is always 1.
$i^1 = i$	Anything to the power of 1 is itself.
$i^2 = -1$	This is true because $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1$
$i^3 = -i$	This is true because $i^3 = i^2 \cdot i = -1 \cdot i = -i$
$i^4 = 1$	This is true because $i^4 = i^3 \cdot i = -i \cdot i = -\sqrt{-1} \cdot \sqrt{-1} = -(-1) = 1$

Notice we get back to 1. This whole pattern will kept repeating. So the pattern repeats every fourth power of i. We can use this principle to answer the following:

EXAMPLE: Simplify: i^{38}.

To do these kind of problems, always divide the power by 4 since our pattern above repeats after every four. You want to get the remainder. If we divide 38 by 4 we will get 9 with 2 left over. Whatever the remainder is will become the new power. So $i^{38} = i^2 = -1$. So our conclusion is that i^{38} simplifies to – 1.

EXAMPLE: Simplify: i^{359}.

Again we divide the power (359) by 4. You will get 89 with 3 left over. Since 3 is our remainder, then 3 becomes our new power: $i^{359} = i^3 = -i$. So our conclusion is that i^{359} simplifies to $-i$.

Standard Form of an Imaginary (Complex) Number

a + bi

Here the a is always the real number part and b represents the imaginary, or complex part. If a question asks you to write your answer is standard form you must always put the real number part first and the i part second.

EXAMPLE: Add and simplify: $(9+12i)+(5-2i)$. Write your answer in standard form.

Here you need to add the like terms. The real numbers are like terms and the i terms are also like terms. Add the real numbers and you will get 14. Add the i terms and you will get 10i. Now just write your answer in standard form: 14 + 10i.

EXAMPLE: Subtract and simplify: $(8-5i)-(-2+6i)$ and write your answer in standard form.

We will first distribute the negative and we will get: $8-5i+2-6i$. Now add the like terms. We can add 8 and 2 to get 10 and we can add -5i and -6i to get -11i. Now write in standard form: 10 – 11i. Again the real number part must come first and the i part second.

EXAMPLE: Multiply and simplify: $(2+5i)(3-4i)$. Write your answer in standard form.

For this one we need to use the FOIL method. This means we multiply the First, Outer, Inner, and Last terms. You will get: $6-8i+15i-20i^2$. We can simplify this to: $6+7i-20i^2$. From our list at the beginning of the section, we know that $i^2=-1$. So we will replace the i^2 in the above expression with -1. Then you will have: $6+7i-20(-1)$. Now simplify and write in standard form. Your answer is $26+7i$.

EXAMPLE: If $z=1-2i$ and $w=3-4i$, find $z\cdot\overline{w}$ and write your answer in standard form.

Notation-wise, $\overline{w}$ means you want to take the conjugate of w. This means we want the opposite sign, so $\overline{w}=3+4i$. So now $z\cdot\overline{w}$ can be written as $(1-2i)(3+4i)$. Just multiply this out using the FOIL method: $3+4i-6i-8i^2$. After simplifying you have: $3-2i-8i^2$. Now put in a -1 for i^2 and you will have $3-2i-8(-1)$ which is $11-2i$ in standard form.

EXAMPLE: Simplify and write your answer in standard form: $\sqrt{4}-\sqrt{-16}$.

We can rewrite this problem as: $\sqrt{4}-\sqrt{16\cdot -1}$. Then we can separate this into: $\sqrt{4}-\sqrt{16}\cdot\sqrt{-1}$. We can put in an i for $\sqrt{-1}$. Then we have $\sqrt{4}-\sqrt{16}\cdot i$. We can take the square roots to get: $2-4i$.

EXAMPLE: Divide and simplify: $\dfrac{4-2i}{3+i}$. Write your answer in standard form.

In order to do this we need to multiply the top and bottom by something so that I don't have an i in the denominator. The conjugate will work here, which is 3 – i.

$\dfrac{4-2i}{3+i}\cdot\dfrac{3-i}{3-i}$ So now that we have it set up we need to multiply across the top and bottom.

$\dfrac{12-4i-6i+2i^2}{9-i^2}$ Across the top is the same process as the last example. On the bottom is a difference of squares: $(a+b)(a-b)=a^2-b^2$. Now we need to simplify the top and bottom.

$\dfrac{12-10i+2i^2}{9-i^2}$ Again we need to replace the i^2 with -1.

$\dfrac{12-10i+2(-1)}{9-(-1)}$ Now simplify.

$\dfrac{10-10i}{10}$ This is not written in standard form. We can divide both things on top by 10.

$1-i$ This is written in standard form and is our answer.

EXAMPLE: Divide and simplify: $\dfrac{\sqrt{4}+\sqrt{-49}}{\sqrt{25}-\sqrt{-9}}$. Write your answer in standard form.

First we need to get rid of the roots. You will get: $\dfrac{2+7i}{5-3i}$.

Now we need to multiply the top and bottom by the conjugate of the bottom, which is 5 + 3i.

$\dfrac{2+7i}{5-3i}\cdot\dfrac{5+3i}{5+3i}$ Now multiply across the top and bottom of the fraction.

$\dfrac{10+6i+35i+21i^2}{25-9i^2}$ We can simplify the numerator.

$\dfrac{10+41i+21i^2}{25-9i^2}$ Now put in a -1 for i^2.

$\dfrac{10+41i+21(-1)}{25-9(-1)}$ Now simplify

$\dfrac{-11+41i}{34}$ This is not in standard form, so divide each thing in the top by 34.

$\dfrac{-11}{34}+\dfrac{41}{34}i$ Now this is in standard form even though we can't simplify the fractions anymore.

EXAMPLE: Simplify and write your answer in standard form: $\sqrt{-12}\left(\sqrt{-4}-\sqrt{2}\right)$.

First we need to change all of these into imaginary numbers: For the $\sqrt{-12}$ we can break this down by writing $\sqrt{4\cdot 3\cdot -1}=2\sqrt{3}\cdot i=2i\sqrt{3}$. Usually we put the i in front of the square root to avoid confusion of whether the i is under the square root or not. Changing everything to imaginary numbers we get: $2i\sqrt{3}\left(2i-\sqrt{2}\right)$. Now we distribute: $4i^2\sqrt{3}-2i\sqrt{6}$. We can replace the i^2 with -1: $-4\sqrt{3}-2i\sqrt{6}$.

EXAMPLE: Solve the equation using the quadratic formula and write in standard form: $0=3x^2-4x+6$.

To review, the quadratic formula is: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ for $ax^2+bx+c=0$. Here a = 3, b = -4, and c = 6.

$x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(3)(6)}}{2(3)}=\dfrac{4\pm\sqrt{16-72}}{6}=\dfrac{4\pm\sqrt{-56}}{6}=\dfrac{4\pm 2i\sqrt{14}}{6}=\dfrac{4}{6}\pm\dfrac{2i\sqrt{14}}{6}=\dfrac{2}{3}\pm i\dfrac{\sqrt{14}}{3}$. Here, we can write $\sqrt{-56}$ as $\sqrt{4*14*-1}=\sqrt{4}*\sqrt{14}*\sqrt{-1}=2i\sqrt{14}$. We needed to divide everything individually by 6 since they wanted us to write this in standard form.