10.7 Plane Curves and Parametric Equations

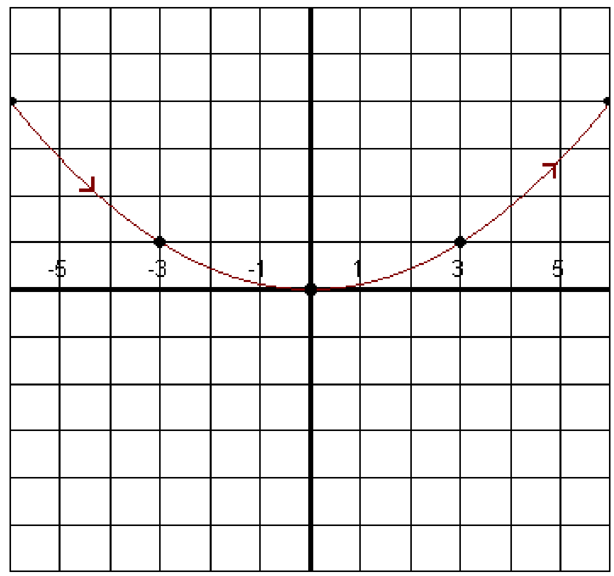
If we take a point (x, y) and move it on the x–y plane after a time t, we have a pair of equations:

$x = f(t)$ and $y = g(t)$

These equations are called **parametric equations** with parameter t.

EXAMPLE: Graph the following equations $x = 3t$ and $y = t^2$ where $-2 \le t \le 2$. Find the rectangular equation.

In order to do this we will make a table and then plot the point. We will start with t = -2 and go to t = 2. You can use any values between -2 and 2. Each value for t will give you an x and a y. When put together this gives us a point to plot.



t	$x = 3t$	$y = t^2$	(x, y)
-2	3(-2) = -6	$(-2)^2 = 4$	(-6, 4)
-1	3(-1) = -3	$(-1)^2 = 1$	(-3, 1)
0	3(0) = 0	$(0)^2 = 0$	(0, 0)
1	3(1) = 3	$(1)^2 = 1$	(3, 1)
2	3(2) = 6	$(2)^2 = 4$	(6, 4)

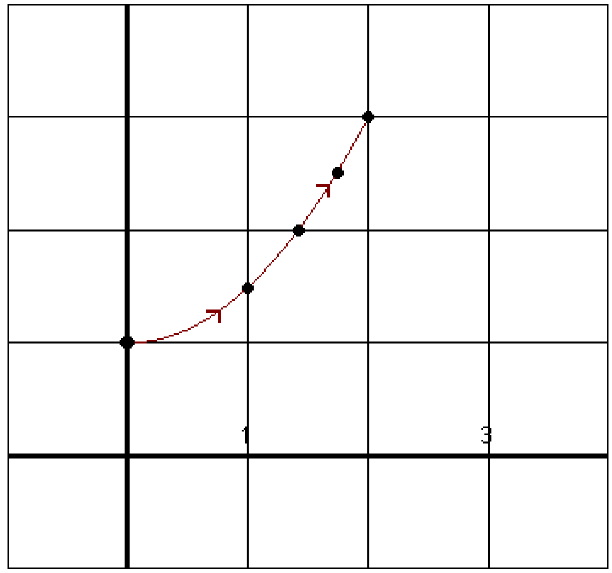
The graph also needs directional arrows. When t is -2 we started at (6, 4). Each time t increases the point moves from left to right, so we indicate that on our graph be arrows.

Now we need to write the rectangular equation. This means you want to write an equation the same as what we graphed that does not contain a t. Usually you want to take one of the equations and solve for t. Then substitute this into the other equation. Our first equation is $x = 3t$. If we solve this for t we get: $t = \frac{x}{3}$. Now we want to substitute this into our other equation for y: $y = \left(\frac{x}{3}\right)^2$. We will get $y = \frac{x^2}{9}$. This is our answer. If we made a table with this equation we would get the same graph as above.

EXAMPLE: Graph the following equations $x = \sqrt{t}$ and $y = \frac{1}{2}t + 1$ where $0 \le t \le 4$. Find the rectangular equation.

In order to do this we will make a table and then plot the points. We will start with t = 0 and go to t = 4. You can use any values between 0 and 4. Each value for t will give you an x and a y. When put together this gives us a point to plot. Make sure you remember to put the correct directional arrows. When t is 0 you are at the point (0, 1). The point is traveling up. To find the rectangular equation, first solve for t in $x = \sqrt{t}$. After squaring both sides we get $t = x^2$. Then we plug this into the y equation to get: $y = \frac{1}{2}x^2 + 1$.

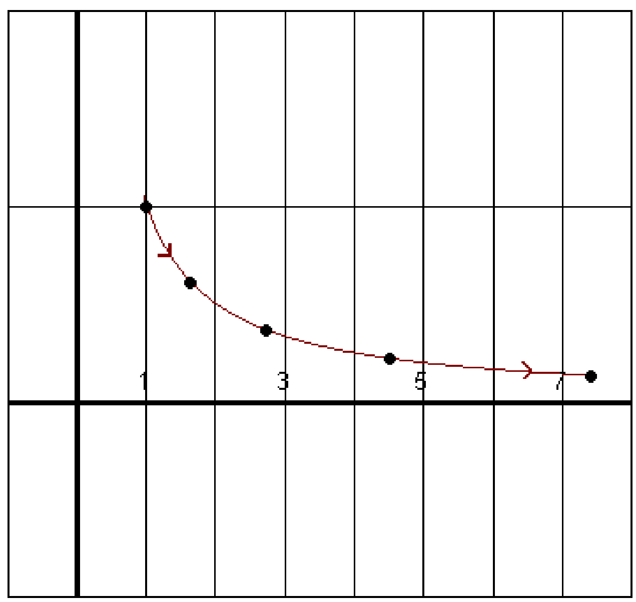
t	$x=\sqrt{t}$	$y=\frac{1}{2}t+1$	(x, y)
0	$x=\sqrt{0}=0$	$\frac{1}{2}(0)+1=1$	(0, 1)
1	$x=\sqrt{1}=1$	$\frac{1}{2}(1)+1=\frac{3}{2}$	$\left(1,\ \frac{3}{2}\right)$
2	$x=\sqrt{2}=1.41$	$\frac{1}{2}(2)+1=2$	(1.41, 2)
3	$x=\sqrt{3}=1.73$	$\frac{1}{2}(3)+1=\frac{5}{2}$	$\left(1.73,\ \frac{5}{2}\right)$
4	$x=\sqrt{4}=2$	$\frac{1}{2}(4)+1=3$	(2, 3)



EXAMPLE: Graph the following equations $x=e^{t}$ and $y=e^{-t}$ where $0\le t\le 2$. Find the rectangular equation.

First we make our table and plot the points. Since t is between 0 and 2 I wanted to do more than just plot three points. That is why I chose 0.5 and 1.5. I wanted to have enough points to plot.

t	$x=e^{t}$	$y=e^{-t}$	(x, y)
0	$x=e^{0}=1$	$x=e^{-0}=1$	(1, 1)
.5	$x=e^{0.5}=1.65$	$x=e^{-0.5}=0.61$	(1.65, 0.61)
1	$x=e^{1}=2.72$	$x=e^{-1}=0.37$	(2.72, 0.37)
1.5	$x=e^{1.5}=4.48$	$x=e^{-1.5}=0.22$	(4.48, 0.22)
2	$x=e^{2}=7.4$	$x=e^{-2}=0.1353$	(7.4, 0.1353)



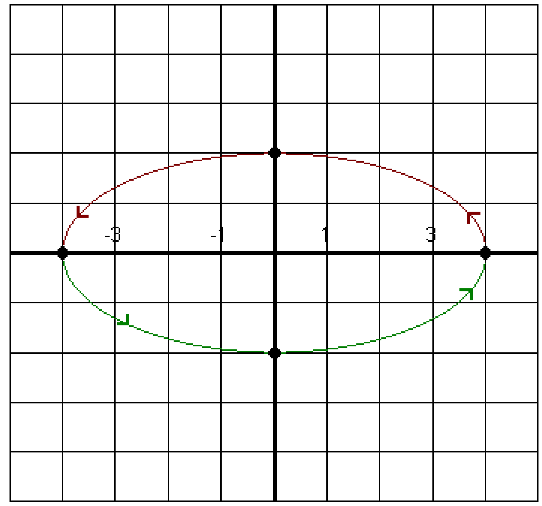
To find the rectangular equation, solve for t in the first equation. In order to do this we must take the natural log of both sides: $\ln x=\ln e^{t}$. We get $t=\ln x$. Now plug this into the second equation for t. You will get: $y=e^{-\ln x}$.

This simplifies to: $y=x^{-1}=\frac{1}{x}$.

EXAMPLE: Graph the following equations $x = 4\cos t$ and $y = 2\sin t$ where $0 \le t \le 2\pi$. Find the rectangular equation.

We make our table. I picked values off the unit circle that would give either a -1, 0, or 1 and an answer.

t	$x = 4\cos t$	$y = 2\sin t$	(x, y)
0	$x = 4\cos 0 = 4(1) = 4$	$y = 2\sin 0 = 2(0) = 0$	(4, 0)
$\frac{\pi}{2}$	$x = 4\cos\frac{\pi}{2} = 4(0) = 0$	$y = 2\sin\frac{\pi}{2} = 2(1) = 2$	(0, 2)
π	$x = 4\cos\pi = 4(-1) = -4$	$y = 2\sin\pi = 2(0) = 0$	(-4, 0)
$\frac{3\pi}{2}$	$x = 4\cos\frac{3\pi}{2} = 4(0) = 0$	$y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$	(0, -2)
2π	$x = 4\cos 2\pi = 4(1) = 4$	$y = 2\sin 2\pi = 2(0) = 0$	(4, 0)

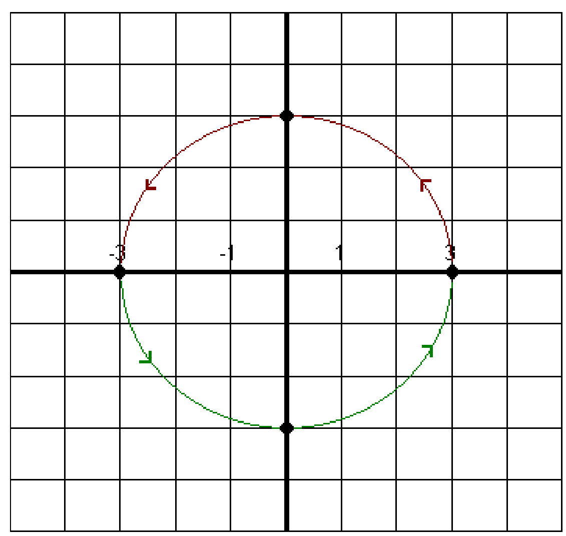


What kind of graph do we get? That's right, it's an ellipse. In order to write the rectangular equation we just need to come up with the equation of this ellipse. It is going horizontally, so we know that a is 4 and b is 2 and this is centered at the origin. The equation is: $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$, or $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

EXAMPLE: Graph the following equations $x = 3\cos t$ and $y = 3\sin t$ where $0 \le t \le 2\pi$. Find the rectangular equation.

We make our table. I picked values off the unit circle that would give either a -1, 0, or 1 and an answer.

t	$x = 3\cos t$	$y = 3\sin t$	(x, y)
0	$x = 3\cos 0 = 3(1) = 3$	$y = 3\sin 0 = 3(0) = 0$	(3, 0)
$\frac{\pi}{2}$	$x = 3\cos\frac{\pi}{2} = 3(0) = 0$	$y = 3\sin\frac{\pi}{2} = 3(1) = 3$	(0, 3)
π	$x = 3\cos\pi = 3(-1) = -3$	$y = 3\sin\pi = 3(0) = 0$	(-3, 0)
$\frac{3\pi}{2}$	$x = 3\cos\frac{3\pi}{2} = 3(0) = 0$	$y = 3\sin\frac{3\pi}{2} = 3(-1) = -3$	(0, -3)
2π	$x = 3\cos 2\pi = 3(1) = 3$	$y = 3\sin 2\pi = 3(0) = 0$	(3, 0)



What kind of graph do we get? That's right, it's a circle. In order to write the rectangular equation we just need to come up with the equation of this circle. It has a radius of 3, centered at the origin, so the equation is: $x^2 + y^2 = 3^2$.

EXAMPLE: Write parametric equations for the curve $y = -3x + 2$ with the definition $x = t$.

We already have one of parametric equations given, and this is $x = t$. Next we can replace the x with t in the original equation to get $y = -3t + 2$. So our parametric equations are $x = t$ and $y = -3t + 2$.

EXAMPLE: Write parametric equations for the curve $y = 6x - 4$ with the definition $x = \frac{t}{3}$.

We already have one of parametric equations given, and this is $x = \frac{t}{3}$. Next we can replace the x with $\frac{t}{3}$ in the original equation to get $y = 6\left(\frac{t}{3}\right) - 4$. So our parametric equations are $x = \frac{t}{3}$ and $y = 2t - 4$.

EXAMPLE: Write parametric equations for the curve $y = 7 - 2x$ with the definition $x = -5t^2$.

We already have one of parametric equations given, and this is $x = -5t$. Next we can replace the x with $-5t^2$ in the original equation to get $y = 7 - 2\left(-5t^2\right)$. So our parametric equations are $x = -5t$ and $y = 10t^2 + 7$.