4.1 Angles

This section will cover how angles are drawn and also arc length and rotations.

Angles are measured a couple of different ways. The first unit of measurement is a **degree** in which 360° (degrees) is equal to one revolution. Most likely the reason why we use 360 is from the Babylonians, whose year is based on 360 days.

Another unit of measurement for angles is **radians**. In radians, 2π is equal to one revolution. So a conversion between radians and degrees is $2\pi = 360^\circ$, or $\pi = 180^\circ$.

When converting from degrees to radians:

Multiply your degrees by $\frac{\pi}{180}$

When converting from radians to degrees:

Multiply your radians by $\frac{180}{\pi}$

EXAMPLE: Convert 60° to radians.

We will take 60 and multiply it by $\frac{\pi}{180}$ and you will get: $60 \cdot \frac{\pi}{180}$. This reduces to $\frac{\pi}{3}$.

EXAMPLE: Convert -405° to radians.

We will take –405 and multiply it by $\frac{\pi}{180}$ and you will get: $-405 \cdot \frac{\pi}{180}$. This reduces to $-\frac{9\pi}{4}$.

EXAMPLE: Convert $\frac{4\pi}{3}$ into degrees.

We will take $\frac{4\pi}{3}$ and multiply it by $\frac{180}{\pi}$ and you will get: $\frac{4\pi}{3} \cdot \frac{180}{\pi}$. This reduces to 240°.

EXAMPLE: Convert $-\frac{3\pi}{2}$ into degrees.

We will take $-\frac{3\pi}{2}$ and multiply it by $\frac{180}{\pi}$ and you will get: $-\frac{3\pi}{2} \cdot \frac{180}{\pi}$. This reduces to -270°

Degrees, Minutes, Seconds

A degree can be divided into 60 equal parts called **minutes** ($\min'$) and each minute is divided into 60 equal parts called **seconds** ($\sec''$). Below are the conversions:

$$1 \text{ min} = \left(\frac{1}{60}\right)^\circ \text{ or } 1' = \left(\frac{1}{60}\right)^\circ$$

$$1 \text{ sec} = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ \text{ or } 1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$$

EXAMPLE: Write 83 degrees, 24 minutes, 13 seconds in the correct notation. Then convert to decimal degrees. Round to 4 decimal places.

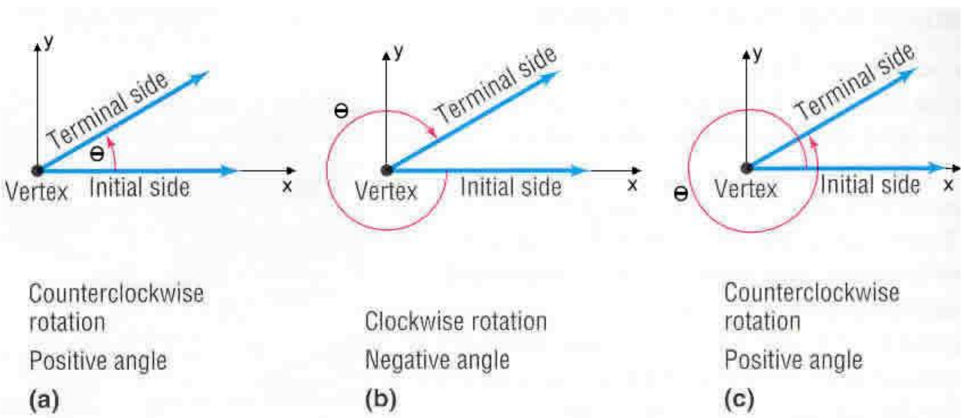
So 83 degrees, 24 minutes, 13 seconds can be written as $83^\circ 24' 13''$. The first part (83) is already in degrees. We just need to change the minute and second portions of the angle into degrees using the conversion factors:

$$83^\circ 24' 13'' = 83^\circ + (24 \text{ min}) \cdot \left(\frac{1^\circ}{60 \text{ min}}\right) + (13 \text{ sec}) \cdot \left(\frac{1^\circ}{3600 \text{ sec}}\right) = 83^\circ + 0.4 + 0.0036\overline{1} = 83.4036^\circ$$

EXAMPLE: Convert 163.36° to degree, minute, second form.

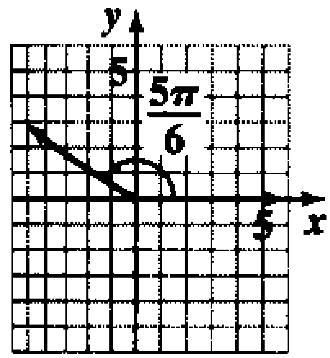
First we split up the angle into a whole number plus the decimal: $163.36^\circ = 163^\circ + 0.36^\circ$. Now convert the decimal part into minutes by using the conversion: $163.36^\circ = 163^\circ + 0.36^\circ \cdot \left(\frac{60'}{1^\circ}\right) = 163^\circ + 21.6'$. Now we will split up the minutes into a whole number and decimal: $21.6' = 21' + 0.6'$. So now we have $163.36^\circ = 163^\circ + 21' + 0.6'$. We need to convert the decimal minute part into seconds by using the conversion: $163.36^\circ = 163^\circ + 21' + 0.6' \cdot \left(\frac{60''}{1'}\right) = 163.36^\circ = 163^\circ + 21' + 36''$. Therefore, $163.36^\circ = 163^\circ 21' 36''$.

We will use θ (theta) to represent an angle's measurement. In the figure below it describes how you know if an angle is positive or negative. The vertex of the angle is at the origin of a rectangular coordinate system. The positive x axis is always where an angle is measure from, and this is called the initial side. An angle drawn this way is said to be in **standard form**. An angle that goes counterclockwise is always positive, and clockwise angles are negative.



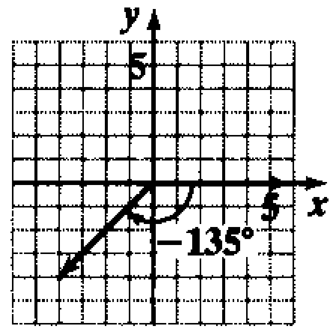
EXAMPLE: Draw each angle in standard position. Indicate which quadrant the angle lies.

a.) $\frac{5\pi}{6}$



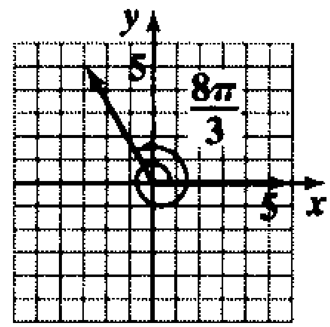
If you are not sure where to draw this angle, first convert it into degrees: $\frac{5\pi}{6} \cdot \frac{180}{\pi} = 150^\circ$. Our angle is measured from the positive x-axis. Since the angle is positive we need to go in the counterclockwise direction. We see that this ends up in Quadrant II.

b.) -135°



Our angle is measured from the positive x-axis. Since we have a negative angle, we need to go in the clockwise direction. We see that we end up in Quadrant III.

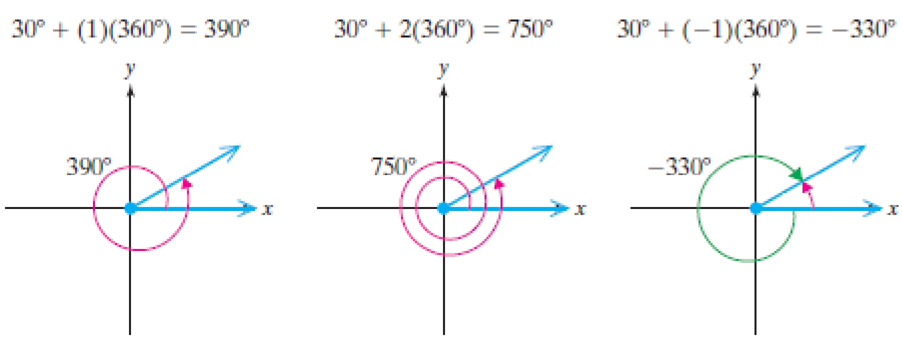
c.) $\frac{8\pi}{3}$



If you are not sure where to draw this angle, first convert it into degrees: $\frac{8\pi}{3} \cdot \frac{180}{\pi} = 480^\circ$. Our angle is measured from the positive x-axis. Since the angle is positive we need to go in the counterclockwise direction. Since this angle is more than 360 degrees, we need to subtract 360 from 480. We will get 120 degrees. So we need to got around 360 degrees and then go an extra 120 degrees. We see that this ends up in Quadrant II.

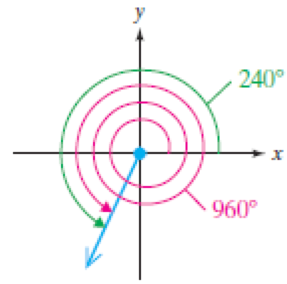
Coterminal Angles

Two angles in standard position with the same initial side and same terminal side are called coterminal angles. The figure below shows three angles in standard position that are coterminal to 30 degrees. Notice that each angle is 30 degrees plus or minus some number of full revolutions clockwise or counterclockwise.



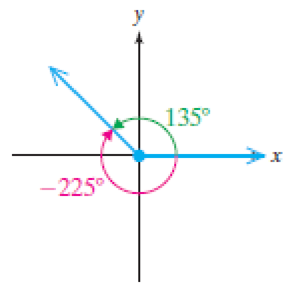
EXAMPLE: Find an angle coterminal to 960° between 0 and 360 degrees. Draw the original angle and the new angle in standard position.

$\theta = 960^\circ$ is more than 360°. Therefore we need to see how many 360°s divide into 960. So if you divide 360 into 960 you will get 2 and a decimal part. So $\theta = 960^\circ - 2(360^\circ) = 240^\circ$. Therefore the angle coterminal to 960° between 0 and 360 degrees is 240°. To the right is both angles drawn in standard position. Notice both angles allow you to arrive at the same position.

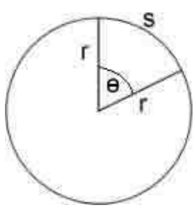


EXAMPLE: Find an angle coterminal to -225° between 0 and 360 degrees. Draw the original angle and the new angle in standard position.

$\theta = -225^\circ$ is less than one revolution. We want to get a positive angle between 0 and 360 degrees. To do this, we need add 360 degrees to our angle. You will get $-225^\circ + 360^\circ = 135^\circ$. Therefore the angle coterminal to -225° between 0 and 360 degrees is 135°. To the right is both angles drawn in standard position. Notice both angles allow you to arrive at the same position.



Arc length – this is the length of the arc between the two lines shown with θ.



The equation is $S = r\theta$, where S is the arc length, r is the radius, and θ MUST be measured in RADIANS! The θ is also called the **central angle**.

EXAMPLE: Find the arc length of a sector whose radius is 3 inches and whose central angle is $\frac{\pi}{3}$.

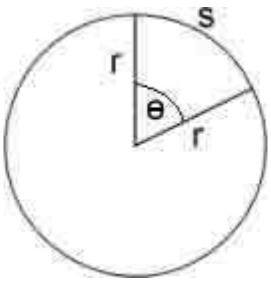
Using our formula $S = r\theta$ we know that $r = 3$ and $\theta = \frac{\pi}{3}$. Putting this into the formula we will get:

$S = 3 \cdot \frac{\pi}{3}$, so $S = \pi$ inches.

EXAMPLE: Find the arc length of a sector whose radius is 7 inches and whose central angle is 45°.

Using our formula $S = r\theta$ we know that r = 7. We will not use 45 degrees as our angle because this is not in degrees. We need to multiply it by $\frac{\pi}{180}$ to get it into radians: $45 \cdot \frac{\pi}{180}$. Reducing this we get $\theta = \frac{\pi}{4}$. Putting this into the formula we will get: $S = 7 \cdot \frac{\pi}{4}$, so $S = \frac{7\pi}{4}$ inches.

Area of a Sector – this is the area of the piece of pie shown below.



The equation is $A = \frac{1}{2}r^2\theta$, where A is the area of the sector, r is the radius, and again θ MUST be measured in RADIANS!

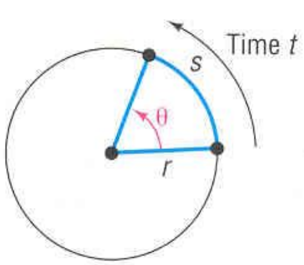
EXAMPLE: Find the area of a sector with a radius of 4 inches with a central angle of $\frac{4\pi}{3}$.

Using our formula $A = \frac{1}{2}r^2\theta$ we know that r = 4 and $\theta = \frac{4\pi}{3}$. Putting this into the formula will give us: $A = \frac{1}{2}4^2\frac{4\pi}{3}$. So we have $A = \frac{1}{2}\cdot\frac{16}{1}\cdot\frac{4\pi}{3}$ which simplifies to $A = \frac{32\pi}{3}$ square inches.

EXAMPLE: Find the area of a sector with a radius of 6 inches with a central angle of 150°.

Using our formula $A = \frac{1}{2}r^2\theta$ we know that r = 6. We will not use 150 degrees as our angle because this is not in degrees. We need to multiply it by $\frac{\pi}{180}$ to get it into radians: $150\cdot\frac{\pi}{180}$. Reducing this we get $\theta = \frac{5\pi}{6}$. Putting this into the formula will give us: $A = \frac{1}{2}6^2\frac{5\pi}{6}$. So we have $A = \frac{1}{2}\cdot\frac{36}{1}\cdot\frac{5\pi}{6}$ which simplifies to $A = 15\pi$ square inches.

Angular Speed (ω)



Angular speed is the angle θ that can be swept out in time t. The formula is:

$\omega = \frac{\theta}{t}$ where θ is in radians.

Linear Speed (v)

This is the speed at which a point on the circle is moving. It is measured by what arc length is traveled in time t. The formula is:

$v = r\omega$ where ω is measured in radians per unit of time.

EXAMPLE: A circular gear rotates at a rate of 75 rpm (revolutions per minute). What is the angular speed (in radians per minute)? What is the linear speed of a point on the gear 3mm from the center?

In order to find the angular speed we need to find the angle in radians. We know that 1 revolution is 2π radians, so we can multiply 75 by 2π to get 150π radians. This is our angle θ. Our unit of time is 1 minute, so we can use the formula $\omega = \frac{\theta}{t}$. We will get $\omega = \frac{150\pi}{1}$ = 150π radians per minute.

For the linear speed we will use $v = r\omega$ with r = 3 and $\omega = 150\pi$. You will get $v = 3 \cdot 150\pi$, so $v = 450\pi$ millimeters per minute.

EXAMPLE: A wheel rotates at a rate of 2160° per second. What is the angular speed (in radians per minute)? What is the linear speed of a point on the wheel 30 cm from the center?

We need to first change 2160° into radians by multiplying by $\frac{\pi}{180}$. You will get: $2160 \cdot \frac{\pi}{180}$ which reduces to 12π radians. Our unit of time is 1 second, so we can use the formula $\omega = \frac{\theta}{t}$. We will get $\omega = \frac{12\pi}{1}$ = 12π radians per second. The question is asking us for radians per minute, so we need to multiply our answer by 60. You will get $\omega = 720\pi$ radians per minute.
Now we can find the linear speed. We will use $v = r\omega$ with r = 30 and $\omega = 720\pi$. You will get $v = 30 \cdot 720\pi$ which is $v = 21600\pi$ centimeters per minute.

EXAMPLE: To approximate the speed of the current of a river, a circular paddle wheel with a radius of 4 feet is lowered into the water. If the current causes the wheel to rotate at a speed of 10 rpm, what is the speed of the current in miles per hour (1 mile = 5280 feet).

We can convert 10rpm to radians per minute by multiplying it by 2π to get 20π radians per minute. This is our angular speed, ω. The water moving is a linear speed, so we need to use the formula $v = r\omega$. We know r = 4 and $\omega = 20\pi$, so $v = 4 \cdot 20\pi$, or $v = 80\pi$ feet per minute. We need to now change this into miles per hour, so we need to use some dimensional analysis. Basically we get the units to cancel by using the appropriate conversions:

$$\frac{80\pi \;\; ft}{1 \;\; \min} \cdot \frac{60 \min}{1 hr} \cdot \frac{1 \;\; mile}{5280 \;\; ft} = 2.86 \;\; mph$$