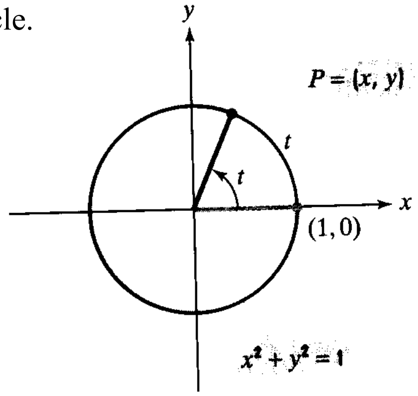
4.2 Trigonometric Functions: Unit Circle Approach

A **unit circle** is a circle centered at the origin with a radius of 1. Its equation is $x^2 + y^2 = 1$ as shown in the drawing below. Here the letter t represents an angle measure. The point P=(x, y) represents a point on the unit circle.



The following definitions are given based on this picture.

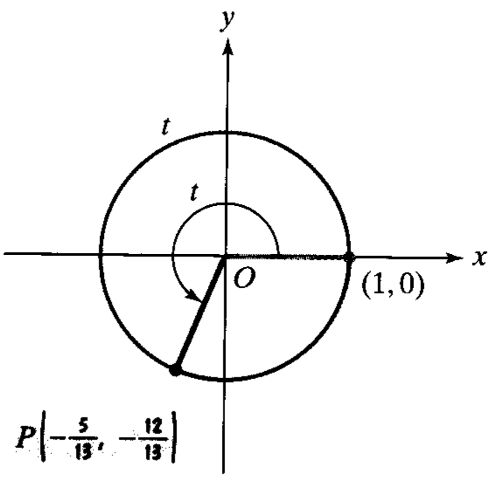
$$\sin t = y \qquad \csc t = \frac{1}{y}$$

$$\cos t = x \qquad \sec t = \frac{1}{x}$$

$$\tan t = \frac{y}{x} \qquad \cot t = \frac{x}{y}$$

If we start with $x^2 + y^2 = 1$ and put in our definitions above we will have $\cos^2\theta + \sin^2\theta = 1$, an identity we will come back to.

EXAMPLE: Suppose a point on the unit circle is $\left(-\frac{5}{13}, -\frac{12}{13}\right)$. Find all six trigonometric values.



We want to use our definitions to answer the questions. We know that $x = -\frac{5}{13}$ and $y = -\frac{12}{13}$ because of the point given. This automatically tells us that $\sin t = -\frac{12}{13}$ and $\cos t = -\frac{5}{13}$. We can plug in x and y into the other equations to find the remaining trig values. $\csc t = \frac{1}{-\frac{12}{13}} = -\frac{13}{12}$ $\quad \sec t = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$

$$\tan t = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{13}\cdot\frac{13}{5} = \frac{12}{5} \qquad \cot t = \frac{-\frac{5}{13}}{-\frac{12}{13}} = \frac{5}{13}\cdot\frac{13}{12} = \frac{5}{12}$$

EXAMPLE: Use a calculator to find the approximate value of $\cos 14^\circ$ rounded to the two decimal places.

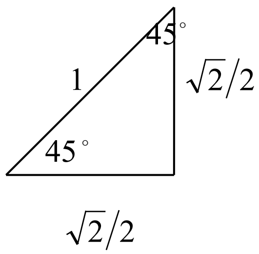
Make sure your calculator is in degree mode, and then enter it into the calculator. You should get 0.97.

EXAMPLE: Use a calculator to find the approximate value of $\sin\frac{\pi}{8}$ rounded to the two decimal places.

You need to have your calculator in radians for this one, because of the π. You should get 0.38.

Now let's look at the angle of $t = 45^\circ$ or $\dfrac{\pi}{4}$. At this angle, we end up with the following triangle:

45 – 45 – 90 Triangle



We can use our definitions of sine, cosine, and tangent to find exact values:

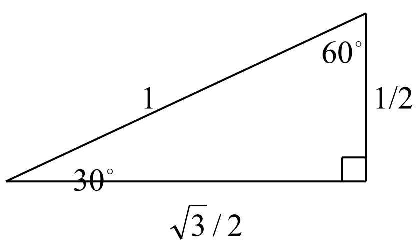
$$\sin 45 = \frac{\sqrt{2}}{2} \qquad \cos 45 = \frac{\sqrt{2}}{2} \qquad \tan 45 = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

From our above definitions we can also find the following:

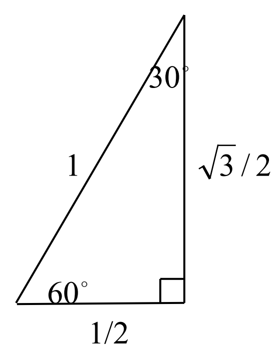
$$\csc 45^\circ = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}\,, \qquad \sec 45^\circ = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}\,, \qquad \cot 45^\circ = \frac{1}{1} = 1$$

Now let's look at two other special angles on the unit circle. We will consider $t = 30^\circ$ and $t = 60^\circ$

30 – 60 – 90 Triangle ($t = 30^\circ$)



30 – 60 – 90 Triangle ($t = 60^\circ$)



We can use our definitions of sine, cosine, and tangent again to find exact values with 30 and 60 degrees.

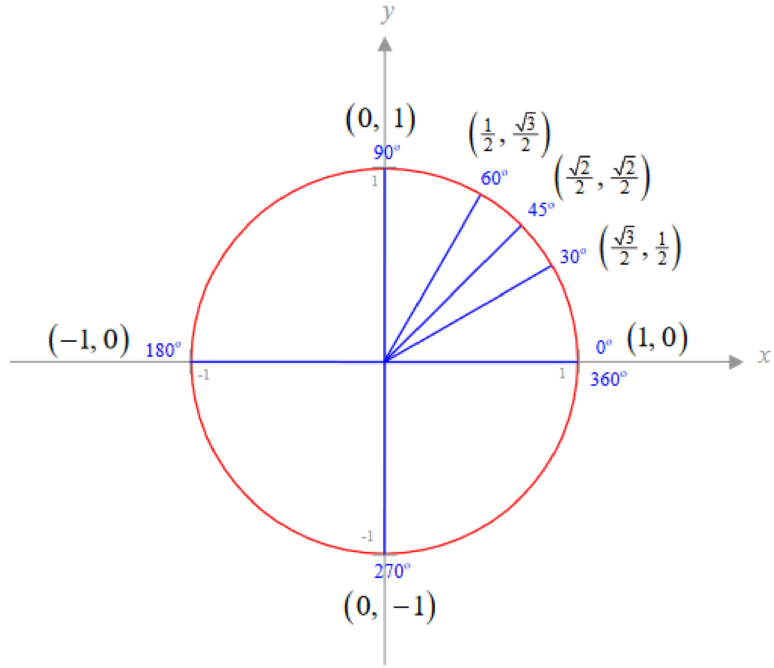
$$\sin 30^\circ = \frac{1}{2} \qquad \cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{1/2}{\sqrt{3}/2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

Table of Trigonometric Values and Unit Circle

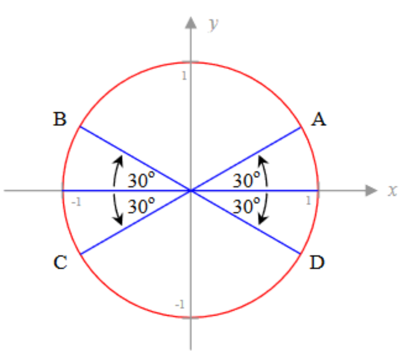
We can display this information in two different ways. The first way shown below is a table. We can also fill in the first quadrant of the unit circle as shown below.

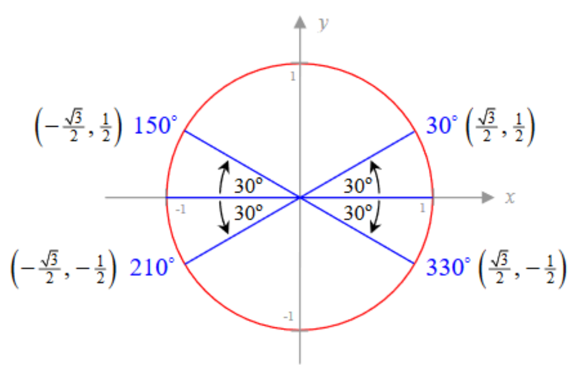
θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined



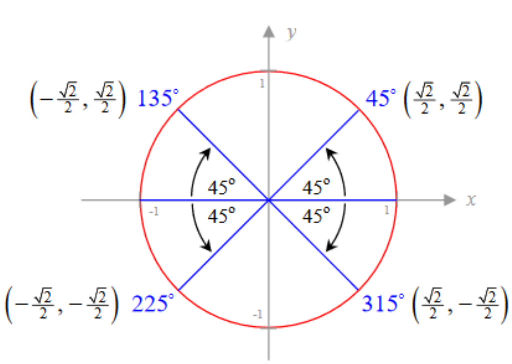
Using Symmetry with the Unit Circle

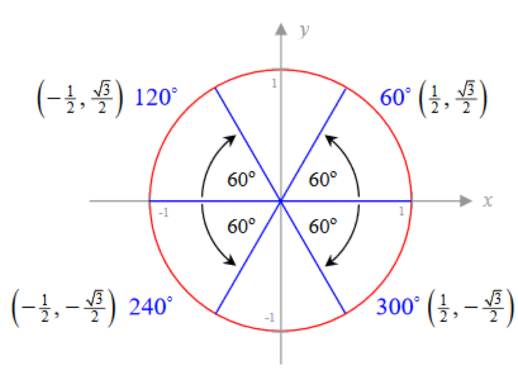
We have the first quadrant of the unit circle filled out, however it would be nice to fill out the rest of the circle. We can do this by symmetry. Below is a diagram that shows where there are other multiples of 30 degrees. As you can see, we can use symmetry to write the coordinates in other quadrants. For example, if we look at 30 degrees and 150 degrees, we see that these two points on the unit circle would have the same y value, however the x value would have opposite signs.



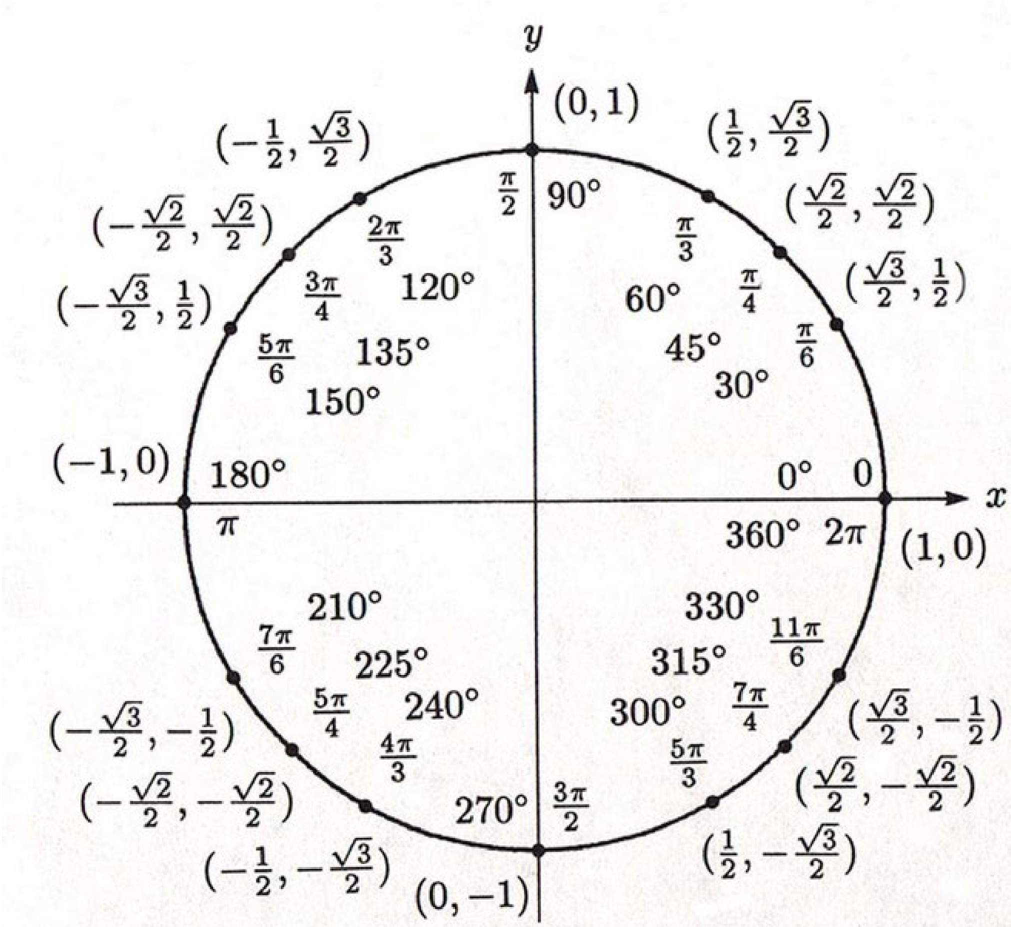


Now we can use this same logic to look at other multiples of 45 degrees and 60 degrees:





Now we can finally put all of this together into to get our unit circle with all special angles labeled:



EXAMPLE: Use the unit circle to evaluate the six trigonometric functions of the real number $t = \frac{2\pi}{3}$.

From our unit circle above, we see that the coordinate is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. From our trig definitions, we can conclude that $\cos t = -\frac{1}{2}$ and $\sin t = \frac{\sqrt{3}}{2}$. We can plug in x and y into the other equations to find the remaining trig values:

$$\csc t = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec t = \frac{1}{-\frac{1}{2}} = -2$$

$$\tan t = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot -\frac{2}{1} = -\sqrt{3} \qquad \cot t = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Even – Odd Properties

$\cos(-t) = \cos t$ $\quad\quad$ $\sec(-t) = \sec t$

$\sin(-t) = -\sin t$ $\quad\quad$ $\csc(-t) = -\csc t$

$\tan(-t) = -\tan t$ $\quad\quad$ $\cot(-t) = -\cot t$

Periodic Properties

If we start at an angle and go around one revolution (360° or 2π radians) we will end up at the same angle we started with. The k value is any integer, and represents how many revolutions are going around. If you want to use degrees, replace the $2\pi k$ in the equations below with $360k$. For tangent and cotangent you will end up at the same spot if you add 2π, however you will also get the same value if you just add π.

$\sin(t \pm 2\pi k) = \sin t$ $\quad\quad$ $\csc(t \pm 2\pi k) = \csc t$

$\cos(t \pm 2\pi k) = \cos t$ $\quad\quad$ $\sec(t \pm 2\pi k) = \sec t$

$\tan(t \pm \pi k) = \tan t$ $\quad\quad$ $\cot(t \pm \pi k) = \cot t$

EXAMPLE: Given that $\sec\left(\frac{11\pi}{12}\right) = \sqrt{2} - \sqrt{6}$, determine the value of $\sec\left(-\frac{13\pi}{12}\right)$.

We want to break up the angle inside in terms of $\frac{11\pi}{12}$. We can write it this way:
$\sec\left(-\frac{13\pi}{12}\right) = \sec\left(\frac{11\pi}{12} - \frac{24\pi}{12}\right)$. We can also say that $\sec\left(-\frac{13\pi}{12}\right) = \sec\left(\frac{11\pi}{12} - 2\pi\right)$. Our Periodic Properties above says that $\sec\left(\frac{11\pi}{12} - 2\pi\right) = \sec\left(\frac{11\pi}{12}\right)$. So since our expression simplifies to $\sec\left(\frac{11\pi}{12}\right)$, we know $\sec\left(-\frac{13\pi}{12}\right) = \sqrt{2} - \sqrt{6}$.

EXAMPLE: Use the Even-Odd Properties and Periodic Properties to simplify:
$-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t)$

For the first term, since we are adding 2π we can say $-2\sin(3t + 2\pi) = -2\sin(3t)$ by applying the Periodic Properties. Next, we can say $-3\sin(-3t) = -3(-\sin(3t)) = 3\sin(3t)$ and $\cos(-2t) = \cos(2t)$ by the Even-Odd Properties.

So $-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t) = -2\sin(3t) + 3\sin(3t) + \cos(2t)$. The two sine terms are like terms, so $-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t) = \sin(3t) + \cos(2t)$.