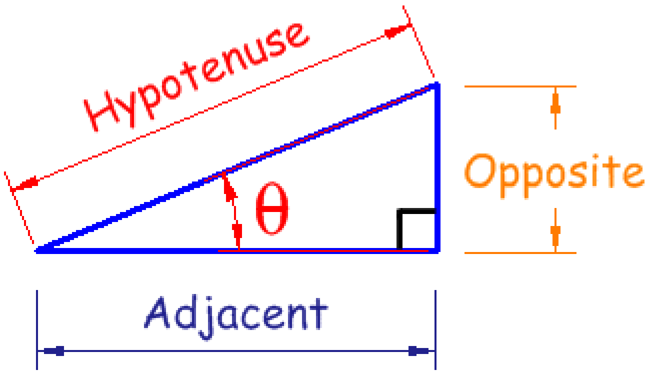
4.3 Right Triangle Trigonometry

This is a very important section since we are giving definitions for the six trigonometric functions you be using throughout the rest of this course and beyond. We need to first start with a drawing of a right triangle. The following definitions only apply to RIGHT TRIANGLES.

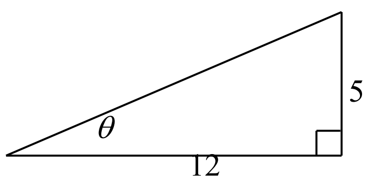


$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \csc\theta = \frac{hypotenuse}{opposite}$$

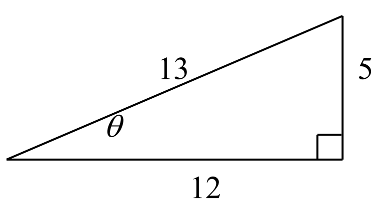
$$\cos\theta = \frac{adjacent}{hypotenuse} \qquad \sec\theta = \frac{hypotenuse}{adjacent}$$

$$\tan\theta = \frac{opposite}{adjacent} \qquad \cot\theta = \frac{adjacent}{opposite}$$

EXAMPLE: Find the exact value of the 6 trig functions using the following figure:



First we need to find the missing side. In this case it is the hypotenuse. In order to find this we need to use the formula $a^2 + b^2 = c^2$. The side opposite the right angle is always side c. So we have $12^2 + 5^2 = c^2$. Simplifying we will get $144 + 25 = c^2$, or $169 = c^2$. We will get $c = \pm 13$, however our answer is c = 13 since we can't have a negative side. Now we are ready to write our 6 trig functions. Here the hypotenuse is 13, opposite is 5, and the adjacent side is 12. We will put these into the formulas and that will be our answers.

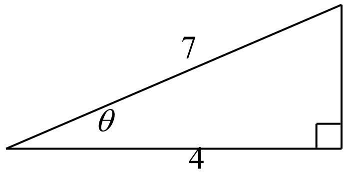


$$\sin\theta = \frac{5}{13} \qquad \csc\theta = \frac{13}{5}$$

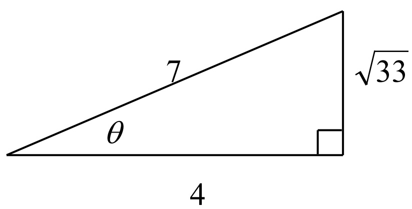
$$\cos\theta = \frac{12}{13} \qquad \sec\theta = \frac{13}{12}$$

$$\tan\theta = \frac{5}{12} \qquad \cot\theta = \frac{12}{5}$$

EXAMPLE: Find the exact value of the 6 trig functions using the following figure:



First we need to find the missing side. In this case it is the hypotenuse. In order to find this we need to use the formula $a^2 + b^2 = c^2$. The side opposite the right angle is always side c, which is 7 in our figure. So we have $4^2 + b^2 = 7^2$. Simplifying we will get $16 + b^2 = 49$, or $33 = c^2$. We will get $c = \sqrt{33}$. Now we are ready to write our 6 trig functions. Here the hypotenuse is 7, opposite is $\sqrt{33}$, and the adjacent side is 4. We will put these into the formulas and that will be our answers. The answer for cosecant is $\frac{7}{\sqrt{33}}$. This needs to be rationalized by multiplying top and bottom by the square root of 33: $\frac{7}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$. You want to always rationalize so that there is no square root in the denominator.

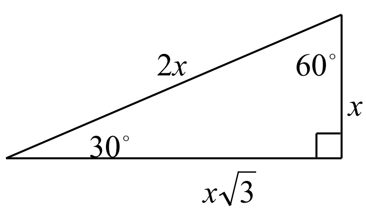


$$\sin\theta = \frac{\sqrt{33}}{7} \qquad \csc\theta = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$$

$$\cos\theta = \frac{4}{7} \qquad \sec\theta = \frac{7}{4}$$

$$\tan\theta = \frac{\sqrt{33}}{4} \qquad \cot\theta = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$$

30 – 60 – 90 Triangle



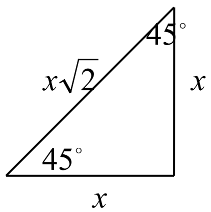
In this triangle the opposite side of 30 degrees is always half of the hypotenuse. The adjacent side is always $\sqrt{3}$ times the opposite. From this relationship we can get values for 30 and 60 degrees.

$$\sin 30 = \frac{x}{2x} = \frac{1}{2} \qquad \sin 60 = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

$$\cos 30 = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2} \qquad \cos 60 = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 30 = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60 = \frac{x\sqrt{3}}{x} = \sqrt{3}$$

45 – 45 – 90 Triangle



In this triangle the opposite and adjacent sides are the same. The hypotenuse is always $\sqrt{2}$ times the opposite or adjacent. From this relationship we can get values for 45 degrees.

$$\sin 45 = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45 = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45 = \frac{x}{x} = 1$$

As with the unit circle, you can put all the information on table, but I will also add the other 3 trig functions:

Table of trigonometric values

θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

EXAMPLE: Find the exact value without using a calculator: $2\cos^2 30^\circ - \sin 30^\circ$

This can be rewritten as: $2(\cos 30^\circ)^2 - \sin 30^\circ$. Now substitute values by using the table.

$2\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2}$. Now square everything inside the parenthesis: $2\left(\frac{3}{4}\right) - \frac{1}{2}$. Reduce this fraction: $\frac{3}{2} - \frac{1}{2}$. After subtracting we get 1.

EXAMPLE: Find the exact value without using a calculator: $\frac{1-\cos 60^\circ}{\sin 60^\circ}$.

We can put in values right away from the table: $\frac{1-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$. We can subtract on the top to get: $\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$. We can flip over the bottom fraction and multiply to get: $\frac{1}{2}\cdot\frac{2}{\sqrt{3}}$ which is $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

EXAMPLE: Find the exact value without using a calculator: $\tan\frac{\pi}{4}+\cot\frac{\pi}{4}$.

From the table we get a 1 for each trig function, so the problem becomes 1 + 1 = 2.

If you refer back to the previous table of trig values, you will notice that you see the same value repeated. This leads to the following identities:

Cofunction Identities

$\sin\theta=\cos(90-\theta)$	$\csc\theta=\sec(90-\theta)$
$\cos\theta=\sin(90-\theta)$	$\sec\theta=\csc(90-\theta)$
$\tan\theta=\cot(90-\theta)$	$\cot\theta=\tan(90-\theta)$

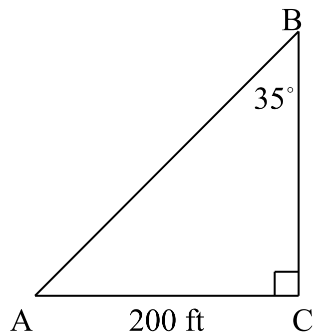
EXAMPLE: Write the following as an equivalent cosine expression: $\sin 33$.

We can use a Cofunction Identity for this. We will use $\sin\theta=\cos(90-\theta)$. Here $\theta=33$. Substituting we get $\sin 33=\cos(90-33)$. So our answer is: $\sin 33=\cos 57$. If you put both of these in your calculator you should get the same decimal.

EXAMPLE: Given the function value, find a cofunction of another angle with the same value: $\csc\frac{\pi}{3}=\frac{2\sqrt{3}}{3}$.

We need to use a Cofunction Identity for this. We will use $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$, the same as above but we are using radians instead of degrees. Here $\theta=\frac{\pi}{3}$. Substituting we get $\csc\frac{\pi}{3}=\sec\left(\frac{\pi}{2}-\frac{\pi}{3}\right)$. So our answer is: $\csc\frac{\pi}{3}=\sec\left(\frac{\pi}{6}\right)$. So we know that $\sec\frac{\pi}{6}$ also equals $\frac{2\sqrt{3}}{3}$.

EXAMPLE: Solve the triangle:

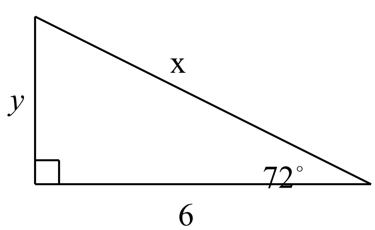


In this problem we need to find side AB, side AC, and $m\angle A$ (measurement of angle A). First we can find $m\angle A$. The sum of all the angles in a triangle is 180 degrees. We have a 35 degree angle and a 90 degree angle, so $m\angle A = 180 - 35 - 90 = 55^\circ$. Now we want to find side AB. Let's call this length x. From our picture above let's use the 35 degree angle. The side opposite this angle is 200 ft. The hypotenuse is x, which is what we are trying to find. We can set up the following equation: $\sin 35^\circ = \dfrac{200}{x}$. We can solve for x by cross multiplying. We get $x = \dfrac{200}{\sin 35^\circ} = 348.69$ ft. Now we can solve for side BC, which we are calling y. Again if we use the 35 degree angle, the y is the adjacent angle and 200 ft is still the opposite side. We will use the following equation to solve for y: $\tan 35^\circ = \dfrac{200}{y}$. We can solve for y by doing cross multiplication:

$y = \dfrac{200}{\tan 35^\circ} = 285.63$ ft. Now we know the measurement of all three sides and angles, so it is solved.

EXAMPLE: A ladder is leaning against a building and forms an angle of 72 degrees with the ground. If the foot of the ladder is 6 feet from the base of the building how far up the building does the ladder reach? How long is the ladder?

First let's draw a picture that describes what is happening. A ladder leaning against a house will give us a right triangle:



This time we want to solve for y. We can do this the same as in the previous problem. We want to find a trig function that relates y (opposite) and 6 (adjacent). This would be tangent. Now we will set up an equation and solve for y, which is the first thing they are asking us to find.

$\tan 72^\circ = \dfrac{y}{6}$ Now cross multiply.

$6\tan 72^\circ = y$ This time we need to use our calculator. Make sure your calculator is in degree mode.

y = 18.47 feet You can just round to 2 decimal places.

For the second question we need to solve for x, which will give us the length of the ladder. We need to use cosine this time since we have an adjacent side and a hypotenuse.

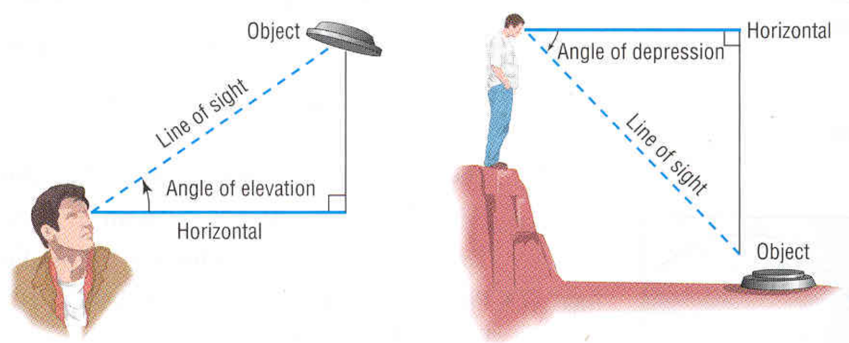
$\cos 72^\circ = \dfrac{6}{x}$ Cross multiply.

$x\cos 72^\circ = 6$ Divide both sides by $\cos 72^\circ$

$x = \dfrac{6}{\cos 72^\circ} = 19.42$ So the length of the ladder is 19.42 feet.

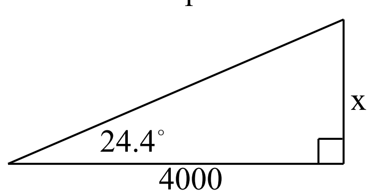
Angle of elevation and depression

When you look up at something you have an angle of elevation, and when you look down on something you have an angle of depression.



EXAMPLE: While standing 4000 feet away from the base of the CN tower in Toronto, the angle of elevation was measured to be 24.4 degrees. Find the height of the tower.

First we need a picture. The angle of elevation is 24.4 degrees, so this is as we look up to the top of the tower. I will call the top of the tower x.



Now we need an equation that relate the opposite side (x) and the adjacent side (4000). This is tangent. So we can set up an equation.

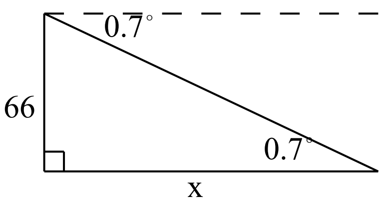
$\tan 24.4^\circ = \dfrac{x}{4000}$ Cross multiply.

$4000\tan 24.4^\circ = x$ Make sure your calculator is in degree mode and solve for x.

$x = 1814.48$ feet

EXAMPLE: An observer in a lighthouse is 66 feet above the surface of the water. The observer sees a ship and finds the angle of depression to be 0.7°. Estimate the distance from the ship to the base of the lighthouse in miles.

We first draw a picture. Notice that the 0.7 degrees is measured from the dotted line since it is an angle of depression. Remember that angles of elevation and depression are always measured from the HORIZONTAL.



From geometry we know that if we have to parallel lines the alternate interior angles are the same. So this is where the 0.7° comes from that is inside the triangle. Now we can use tangent again since it relates the opposite and adjacent sides.

$\tan 0.7^\circ = \dfrac{66}{x}$ Cross multiply.

$x \tan 0.7^\circ = 66$ Divide both sides by $\tan 0.7^\circ$.

$x = \dfrac{66}{\tan 0.7^\circ} = 5401.90 \text{ feet}$ This is in feet, so we need to convert it. We know 1 mile = 5280 feet.

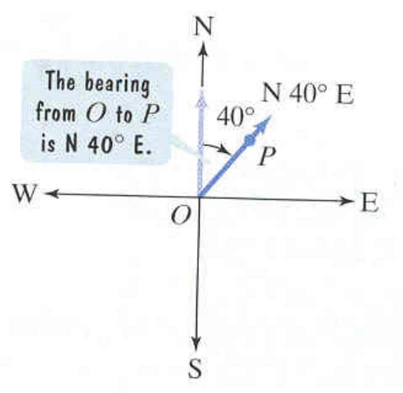
$$\frac{5401.90 \ \textit{ft}}{1} \cdot \frac{1 \ \textit{mile}}{5280 \ \textit{ft}} = 1.02 \ \textit{miles}$$

Bearing

Bearing is a way to measure direction. There are three parts to bearing. The first part is either a N or S. The second part is an acute angle that is ALWAYS measured from a vertical axis. The third part is an E or W. The picture to the right shows how North, South, East, and West is orientated. The next few examples show how to draw bearings.

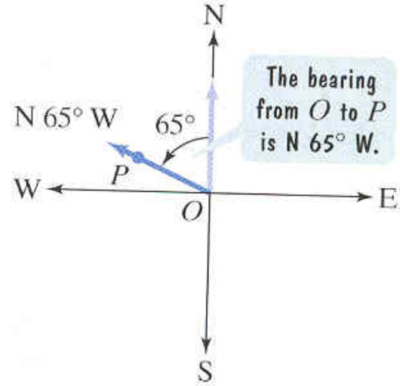
EXAMPLE: Draw the following: $N 40^\circ E$.

North is always straight up. First we go straight north. Then is says to go 40 degrees to the east. So from the north we will measure 40 degrees. Our picture is:



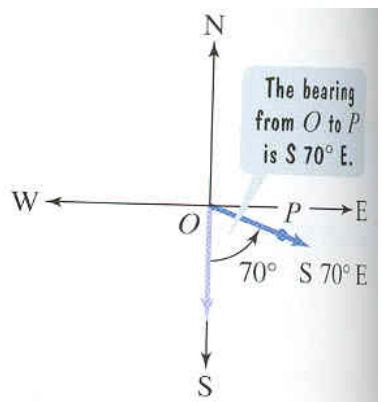
EXAMPLE: Draw the following: $N65^\circ W$.

North is always straight up. First we go straight north. Then is says to go 65 degrees to the west. So from the north we will measure 65 degrees. Our picture is:



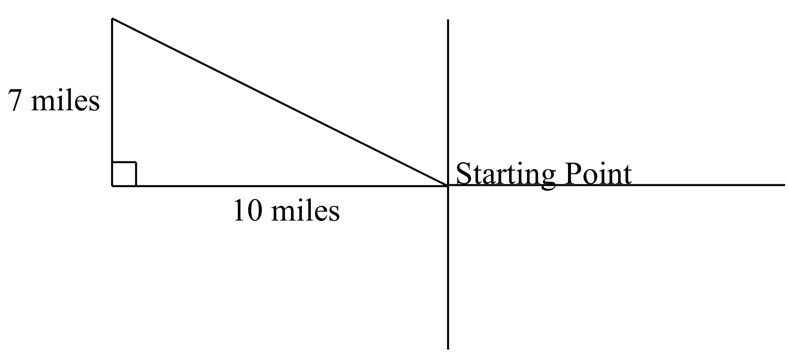
EXAMPLE: Draw the following: $S70^\circ E$.

South is always straight down. First we go straight south. Then is says to go 65 degrees to the east. So from the south we will measure 70 degrees. Our picture is:



EXAMPLE: A cyclist rides west for 10 miles and then north 7 miles. What is the bearing from her starting point?

First let's draw a picture. You will have a right triangle, and we will put the starting point at the origin. Going west from the starting point means we will go left, and label the horizontal side of the triangle as 10 miles. Then the vertical side we will go up 7 miles:



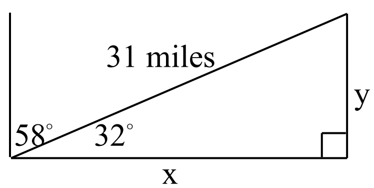
We want to find the angle inside the triangle so we want to set up an equation using a trig definition. We want to pick a trig function that relates the two sides we are given. We want to use tangent:

$\tan\theta = \frac{7}{10}$ Take the inverse tangent of both sides to get $\tan^{-1}\frac{7}{10} \approx 35^\circ$.

But now we want to write this as a bearing. This is the angle inside the triangle, but remember that bearing is measured from the north or south. Therefore we want to find the measure of the angle outside the triangle. This is $90^\circ - 35^\circ = 55^\circ$. Therefore we can write our bearing: $N55^\circ W$.

EXAMPLE: A jeep leaves its present location and travels along a bearing of $N58^\circ E$ for 31 miles. How far north and east of its original position is it? Round to two decimal places as needed.

As before, let's draw a picture. Because we are going north and east, we will draw a triangle in the first quadrant. We will label our bearing angle on the outside of the triangle since the 58 degrees is measured from the north. We find the inside angle by taking 90 degrees – 58 degrees = 32 degrees. So now we want to find the horizontal and vertical sides of the triangle (labeled x and y) by using trig definitions as shown below:



$\sin 32^\circ = \frac{y}{31}$, so $y = 31\sin 32^\circ \approx 16.42$ miles. This is how far north it is from its original position.

$\cos 32^\circ = \frac{x}{31}$, so $x = 31\cos 32^\circ \approx 26.29$ miles. This is how far east it is from the original position.