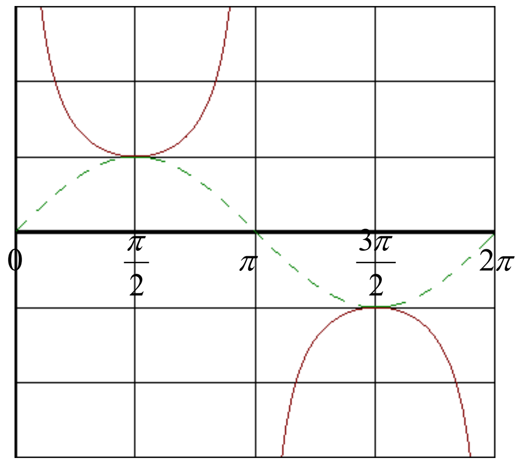
4.6 Graphs of the Other Trigonometric Functions

Now we will look at other types of graphs: $\tan x$, $\cot x$, $\csc x$, $\sec x$. We will start with the cosecant and secant since it follows from the previous section.

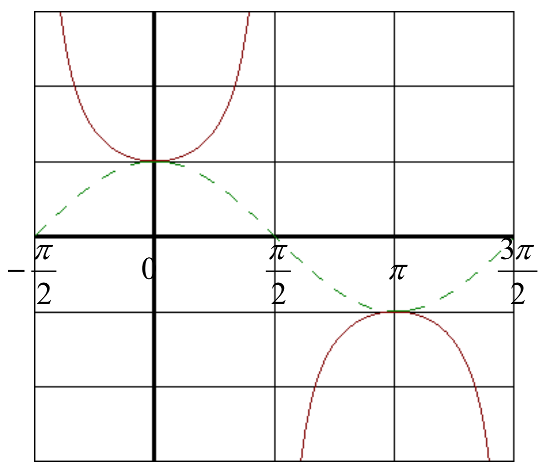
Graph of $y = \csc x$



In order to draw this graph we will first start with the graph of $y = \sin x$ (dotted line). Wherever the graph $y = \sin x$ crosses the x-axis is where there is a vertical asymptote.

The period of $y = \csc x$ is 2π. The amplitude is 1, since the graph touches $y = \sin x$ at its amplitude.

Graph of $y = \sec x$



In order to draw this graph we will first start with the graph of $y = \cos x$ (dotted line). Wherever the graph $y = \cos x$ crosses the x-axis is where there is a vertical asymptote.

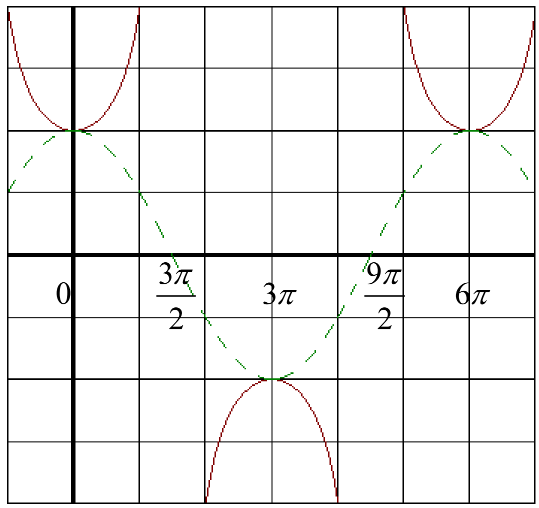
The period of $y = \sec x$ is 2π. The amplitude is 1, since the graph touches $y = \cos x$ at its amplitude.

EXAMPLE: Graph $y = 2\sec\left(\frac{x}{3}\right)$ over one period.

We will first pretend this is a cosine function. The period of this is: Period $= \frac{2\pi}{B}$. In this problem, the period is $\frac{2\pi}{1/3} = 6\pi$. To find the phase shift, we can rewrite our problem as $y = 2\sec\left(\frac{1}{3}x + 0\right)$. Then when we apply the formula for the phase shift, we get zero. This means the graph does not move left or right. Next we can find the increment, which is the period divided by 4: In this case our increment is $\frac{6\pi}{4} = \frac{3\pi}{2}$. We start from zero and keep adding the increment as we did in the previous section.

$0 + \frac{3\pi}{2} = \frac{3\pi}{2}$ $\qquad \frac{3\pi}{2} + \frac{3\pi}{2} = \frac{6\pi}{2}$ $\qquad \frac{6\pi}{2} + \frac{3\pi}{2} = \frac{9\pi}{2}$ $\qquad \frac{9\pi}{2} + \frac{3\pi}{2} = \frac{12\pi}{2}$

So now we have all of our key points, so we can put them all on the graph. Next, we will draw a dotted line to represent the cosine function. Where ever the cosine graph crosses the x-axis is where there will be a vertical asymptote. So you can draw a vertical dotted line through each place it hits the x-axis. Finally we will draw in the curves for the secant function. The final graph will look like this:

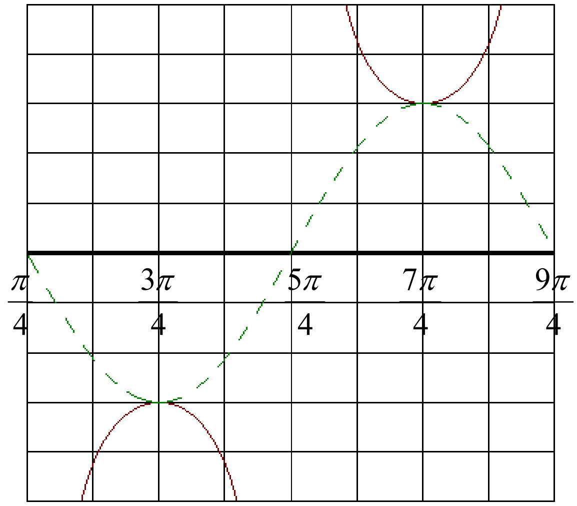


EXAMPLE: Graph $y = -3\csc\left(x - \frac{\pi}{4}\right)$ over one period.

We will first pretend this is a sine function. The period of this is: Period = $\frac{2\pi}{B}$. In this problem, the period is $\frac{2\pi}{1} = 2\pi$. To find the phase shift, we take the opposite sign of c and divide it by b: $\frac{\pi/4}{1} = \frac{\pi}{4}$. This will be our first key point. Next we can find the increment, which is the period divided by 4: In this case our increment is $\frac{2\pi}{4}$. I will leave it in the unreduced form to make it easier to add since you will have common denominators already. We start from $\frac{\pi}{4}$ and keep adding the increment as we did in the previous section.

$\frac{\pi}{4} + \frac{2\pi}{4} = \frac{3\pi}{4}$ $\qquad \frac{3\pi}{4} + \frac{2\pi}{4} = \frac{5\pi}{4}$ $\qquad \frac{5\pi}{4} + \frac{2\pi}{4} = \frac{7\pi}{4}$ $\qquad \frac{7\pi}{4} + \frac{2\pi}{4} = \frac{9\pi}{4}$

So now we have all of our key points, so we can put them all on the graph. Next, we will draw a dotted line to represent the sine function. Where ever the sine graph crosses the x-axis is where there will be a vertical asymptote. So you can draw a vertical dotted line through each place it hits the x-axis. Finally we will draw in the curves for the secant function. The final graph will look like this:

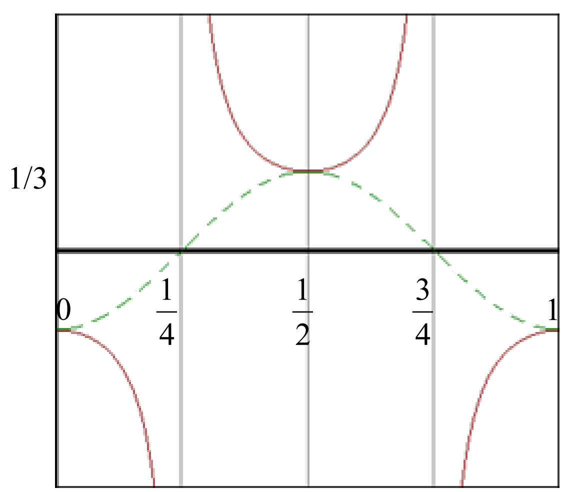


EXAMPLE: Graph $y = -\frac{1}{3}\sec(2\pi x)$ over one period.

We will first pretend this is a cosine function. The period of this is: Period = $\frac{2\pi}{B}$. In this problem, the period is $\frac{2\pi}{2\pi} = 1$. To find the phase shift, we can rewrite our problem as $y = -\frac{1}{3}\sec(2\pi x + 0)$. Then when we apply the formula for the phase shift, we get zero. This means the graph does not move left or right. Next we can find the increment, which is the period divided by 4: In this case our increment is $\frac{1}{4}$. We start from zero and keep adding the increment as we did in the previous section.

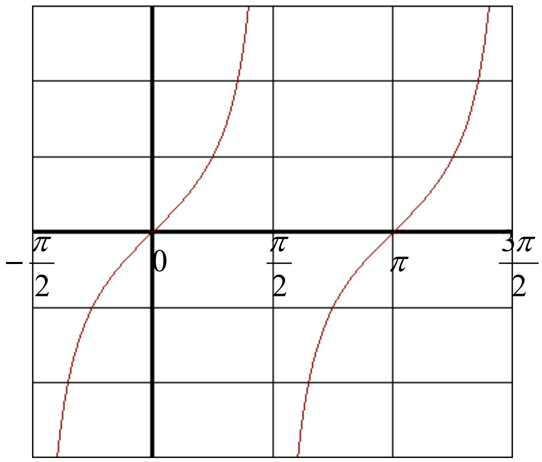
$0 + \frac{1}{4} = \frac{1}{4}$ $\qquad$ $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\qquad$ $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $\qquad$ $\frac{3}{4} + \frac{1}{4} = 1$

So now we have all of our key points, so we can put them all on the graph. Next, we will draw a dotted line to represent the cosine function. Where ever the cosine graph crosses the x-axis is where there will be a vertical asymptote. So you can draw a vertical dotted line through each place it hits the x-axis. Finally we will draw in the curves for the secant function. The final graph will look like this:



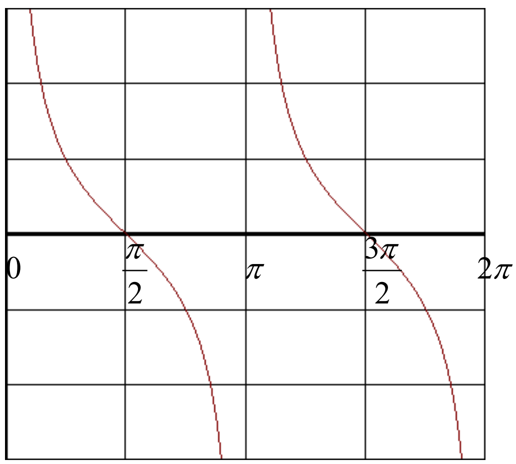
Now we will look at the tangent and cotangent graphs.

Graph of $y = \tan x$



The period is π. There are vertical asymptotes at $x = \frac{n\pi}{2}$ where n is any integer.

Graph of $y = \cot x$



The period is π. There are vertical asymptotes at $x = n\pi$ where n is any integer. Notice the graph goes the opposite direction as the tangent graph.

General Form of a Tangent or Cotangent Equation:

$y = A\tan(Bx - C)$ or $y = A\cot(Bx - C)$

Period $= \frac{\pi}{B}$ **Half point** $= \frac{period}{2}$. The A value is a vertical stretch. Tangent and cotangent graphs do not have an amplitude.

Phase Shift for tangent $= \frac{C}{B} - \frac{\pi}{2B}$ where $y = A\tan(Bx - C)$

Phase Shift for cotangent $= \frac{C}{B}$ where $y = A\cot(Bx - C)$

EXAMPLE: Find the period and phase shift for $y = 3\tan\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

$$\text{Period} = \frac{\pi}{B} = \frac{\pi}{\frac{1}{2}} = 2\pi$$

$$\text{Phase shift} = \frac{C}{B} - \frac{\pi}{2B} = -\frac{\frac{\pi}{3}}{\frac{1}{2}} - \frac{\pi}{2\cdot\frac{1}{2}} = -\frac{2\pi}{3} - \pi = -\frac{5\pi}{3}$$

EXAMPLE: Find the period and phase shift for $y = -2\cot\left(\frac{3\pi}{4}x - \frac{\pi}{6}\right)$

$$\text{Period} = \frac{\pi}{B} = \frac{\pi}{\frac{3\pi}{4}} = \pi\cdot\frac{4}{3\pi} = \frac{4}{3}$$

$$\text{Phase shift} = \frac{C}{B} = \frac{\frac{\pi}{6}}{\frac{3\pi}{4}} = \frac{\pi}{6}\cdot\frac{4}{3\pi} = \frac{2}{9}$$

EXAMPLE: Graph $y = \tan\left(x + \frac{\pi}{4}\right)$ over 2 periods.

First, we will find the period, phase shift, and half-point using the given formulas.

$$\text{Period} = \frac{\pi}{B} = \frac{\pi}{1} = \pi, \quad \text{Half point} = \frac{period}{2} = \frac{\pi}{2} = \frac{2\pi}{4}$$

$$\text{Phase shift} = \frac{C}{B} - \frac{\pi}{2B} = -\frac{\frac{\pi}{4}}{1} - \frac{\pi}{2\cdot 1} = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$$

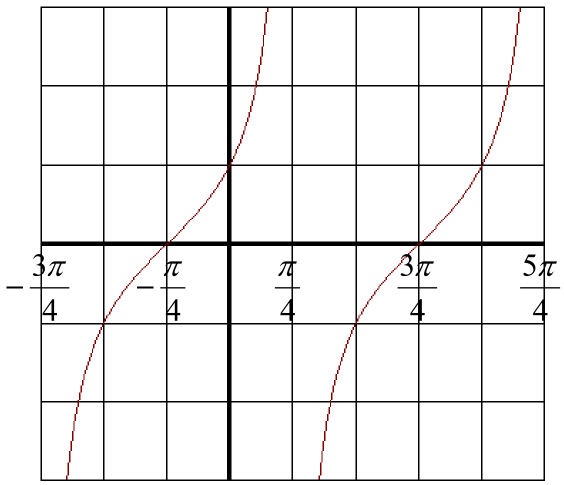
The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift, and add the half-point four times in order to get two periods. These are all the key points. The first one is the phase shift. Notice I made the half-point have a denominator of 4 to make it easier to add:

P.S.

$-\frac{3\pi}{4}$ $\quad -\frac{3\pi}{4}+\frac{2\pi}{4}=-\frac{\pi}{4}$ $\quad -\frac{\pi}{4}+\frac{2\pi}{4}=\frac{\pi}{4}$ $\quad \frac{\pi}{4}+\frac{2\pi}{4}=\frac{3\pi}{4}$ $\quad \frac{3\pi}{4}+\frac{2\pi}{4}=\frac{5\pi}{4}$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at $-\frac{3\pi}{4}$. Then we mark each of the key points. The graph will look like our original graph of tangent. It will always be increasing. It will have the same shape as well since it is not being stretched.

The graph always has this pattern: asymptote, intercept, asymptote, intercept, asymptote.



EXAMPLE: Graph $y=4\tan\left(\frac{\pi}{3}x\right)$ over 2 periods.

First, we will find the period, phase shift, and half-point using the given formulas.

Period $=\frac{\pi}{B} = \frac{\pi}{\frac{\pi}{3}} = \frac{\pi}{1}\cdot\frac{3}{\pi} = 3$, Half point $=\frac{period}{2} = \frac{3}{2}$

Phase shift $= \frac{C}{B}-\frac{\pi}{2B} = \frac{0}{1}-\frac{\pi}{2\cdot\frac{\pi}{3}} = -\frac{\pi}{\frac{2\pi}{3}} = -\frac{\pi}{1}\cdot\frac{3}{2\pi} = -\frac{3}{2}$

The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift, and add the half-point four times in order to get two periods. These are all the key points, including the phase shift which is the first key point:

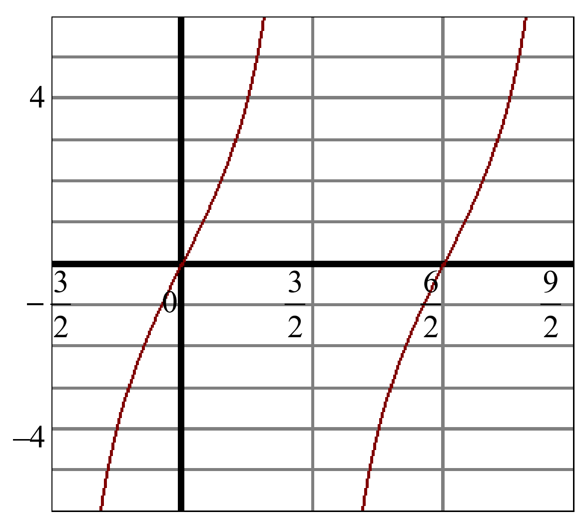
P.S.

$-\frac{3}{2}$ $\quad -\frac{3}{2}+\frac{3}{2}=0$ $\quad 0+\frac{3}{2}=\frac{3}{2}$ $\quad \frac{3}{2}+\frac{3}{2}=\frac{6}{2}$ $\quad \frac{6}{2}+\frac{3}{2}=\frac{9}{2}$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at 0. Then we mark each of the key points. The graph will look like the original graph and will always be increasing.

What does that 4 do? This will stretch the graph vertically. As you can see in the graph below that in between each of the key points, the graph crosses through either 4 or –4. For example, right between $-\frac{3}{2}$ and 0, we see the graph has a y-value of –4. To find the middle point, we can add $-\frac{3}{2}$ and 0 and then divide the result by 2 to get $-\frac{3}{4}$. For the next middle point we will add 0 and $\frac{3}{2}$ and then divide the result by 2 to get $\frac{3}{4}$. For the third middle point we will add $\frac{3}{2}$ and $\frac{6}{2}$ and divide the result by 2 to get $\frac{9}{4}$. For the last middle point we will add $\frac{6}{2}$ and $\frac{9}{2}$ and divide the result by 2 to get $\frac{15}{4}$.

Because of the amplitude, the graph will go through these points: $\left(-\frac{3}{4},-4\right)$, $\left(\frac{3}{4},4\right)$, $\left(\frac{9}{4},-4\right)$, $\left(\frac{15}{4},4\right)$.



EXAMPLE: Graph $y = -3\tan\left(3x - \frac{\pi}{2}\right)$ over 2 periods.

First, we will find the period, phase shift, and half-point.

$$\text{Period} = \frac{\pi}{B} = \frac{\pi}{3}, \quad \text{Half point} = \frac{period}{2} = \frac{\frac{\pi}{3}}{2} = \frac{\pi}{6}$$

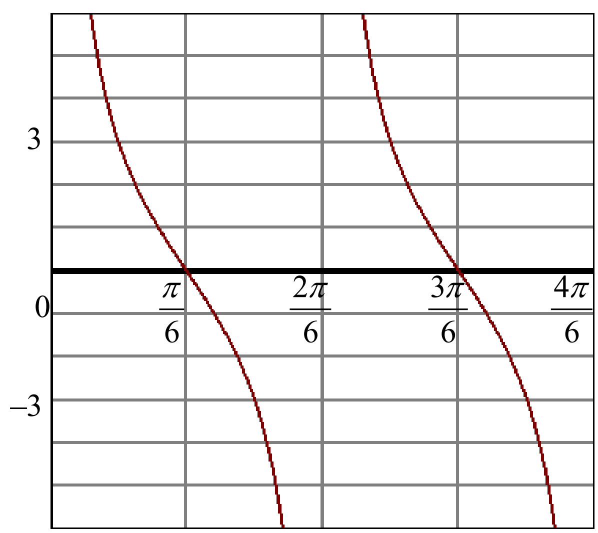
$$\text{Phase shift} = \frac{C}{B} - \frac{\pi}{2B} = \frac{\frac{\pi}{2}}{3} - \frac{\pi}{2\cdot 3} = \frac{\pi}{6} - \frac{\pi}{6} = 0$$

The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift and add the half-point four times in order to get two periods. These are the other key points. I did not reduce in order to make it easier to add:

P.S.				
0	$0 + \frac{\pi}{6} = \frac{\pi}{6}$	$\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}$	$\frac{2\pi}{6} + \frac{\pi}{6} = \frac{3\pi}{6}$	$\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at 0. Then we mark each of the key points. Because there is a negative in front of the graph, the graph will be flipped. So instead of always increasing, the graph will always be decreasing. The -3 out front stretches the graph. As a result, the graph will go through these points: $\left(\frac{\pi}{12}, 3\right)$, $\left(\frac{\pi}{4}, -3\right)$, $\left(\frac{5\pi}{12}, 3\right)$, $\left(\frac{7\pi}{12}, -3\right)$.

As before, the graph will have this pattern: asymptote, intercept, asymptote, intercept, asymptote.



EXAMPLE: Graph $y = -\cot\left(x - \frac{\pi}{2}\right)$ over two periods.

First, we will find the period, phase shift, and half-point.

Period $= \frac{\pi}{B} = \frac{\pi}{1} = \pi$, Half point $= \frac{period}{2} = \frac{\pi}{2}$

Phase shift $= \frac{C}{B} = \frac{\frac{\pi}{2}}{1} = \frac{\pi}{2}$

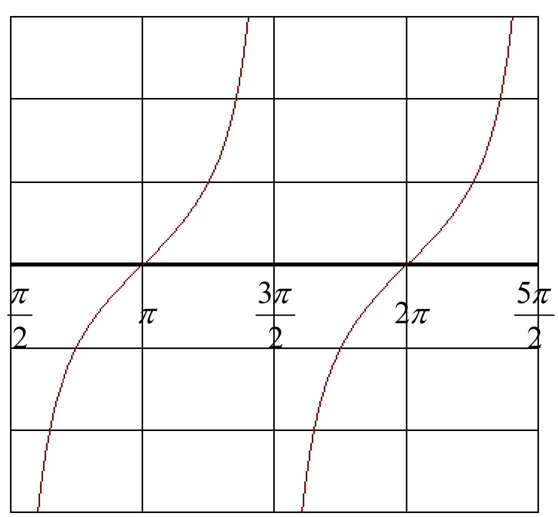
The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift and add the half-point four times in order to get two periods. These are all key points which include the phase shift.

P.S.

$\frac{\pi}{2}$ $\qquad \frac{\pi}{2}+\frac{\pi}{2}=\frac{2\pi}{2}$ $\qquad \frac{2\pi}{2}+\frac{\pi}{2}=\frac{3\pi}{2}$ $\qquad \frac{3\pi}{2}+\frac{\pi}{2}=\frac{4\pi}{2}$ $\qquad \frac{4\pi}{2}+\frac{\pi}{2}=\frac{5\pi}{2}$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at $\frac{\pi}{2}$. Then we mark each of the key points. Because there is a negative in front of the graph, the graph will be flipped. So instead of always decreasing, the graph will always be increasing. Therefore, this will look like a tangent graph. The A value is 1, so graph will not be stretched and will resemble the original shape. However, we can still find the middle points: $\left(\frac{3\pi}{4},-1\right)$, $\left(\frac{5\pi}{4},1\right)$, $\left(\frac{7\pi}{4},-1\right)$, $\left(\frac{9\pi}{4},1\right)$.

As before, the graph will have this pattern: asymptote, intercept, asymptote, intercept, asymptote.



EXAMPLE: Graph $y = \cot\left(\frac{1}{2}x\right)$ over two periods.

First, we will find the period, phase shift, and half-point.

Period = $\frac{\pi}{B} = \frac{\pi}{\frac{1}{2}} = 2\pi$, Half point = $\frac{period}{2} = \frac{2\pi}{2} = \pi$

Phase shift = $\frac{C}{B} = \frac{0}{1} = 0$

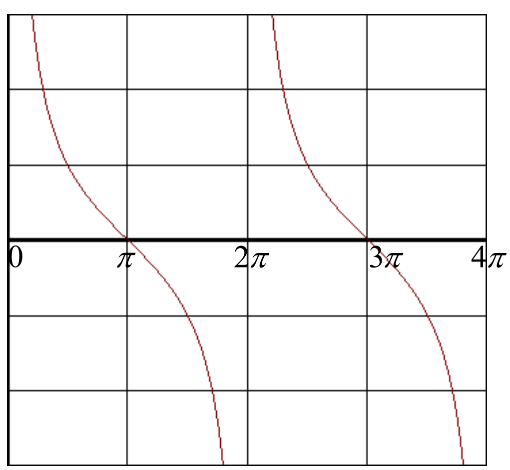
The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift and add the half-point four times in order to get two periods. These are all the key points which include the phase shift.

P.S.

$0 \qquad 0+\pi=\pi \qquad \pi+\pi=2\pi \qquad 2\pi+\pi=3\pi \qquad 3\pi+\pi=4\pi$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at 0. Then we mark each of the key points. Because there is no negative in front of the graph, the graph will not be flipped. The amplitude is 1, so graph will not be stretched and will resemble the original shape. However we can still find the middle points: $\left(\frac{\pi}{2},1\right), \left(\frac{3\pi}{2},-1\right), \left(\frac{5\pi}{2},1\right), \left(\frac{7\pi}{2},-1\right)$.

As before, the graph will have this pattern: asymptote, intercept, asymptote, intercept, asymptote.



EXAMPLE: Graph $y = -2\cot\left(2x + \frac{\pi}{3}\right)$ over 2 periods.

First, we will find the period, amplitude, phase shift, and half-point. I will get common denominators with the half-point and the phase shift to make them easier to add.

Period = $\frac{\pi}{B} = \frac{\pi}{2}$, Amplitude = $|-2| = 2$, Half point = $\frac{period}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}\cdot\frac{3}{3} = \frac{3\pi}{12}$

Phase shift $= \frac{C}{B} = -\frac{\frac{\pi}{3}}{2} = -\frac{\pi}{6}\cdot\frac{2}{2} = -\frac{2\pi}{12}$

The phase shift is where the first asymptote is. Now we need to find the intercepts and the other asymptotes. We do this by using the half-point. We start with the phase shift and add the half-point four times in order to get two periods. These are all the key points which include the phase shift:

P.S.

$-\frac{2\pi}{12}$ $\quad -\frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{12}$ $\quad \frac{\pi}{12} + \frac{3\pi}{12} = \frac{4\pi}{12}$ $\quad \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{7\pi}{12}$ $\quad \frac{7\pi}{12} + \frac{3\pi}{12} = \frac{10\pi}{12}$

To draw the graph, first we start with the phase shift. Therefore, there will be an asymptote at $-\frac{2\pi}{12}$. Then we mark each of the key points. Because there is a negative in front of the graph, the graph will be flipped. The amplitude stretches the graph. As a result of the amplitude, the graph will go through these points: $\left(-\frac{\pi}{24}, -2\right), \left(\frac{5\pi}{24}, 2\right), \left(\frac{11\pi}{24}, -2\right), \left(\frac{17\pi}{24}, 2\right)$.

As before, the graph will have this pattern: asymptote, intercept, asymptote, intercept, asymptote.