4.7 Inverse Trigonometric Functions

From our tables in a previous section we know that $\sin 30^\circ = \frac{1}{2}$. We put in an angle and get a value as a result.

In inverse trig functions we put in the value and get an angle: $\sin^{-1}\frac{1}{2} = 30^\circ$. So here we put in the value of one half and got 30 degrees as a result. We are not allowed to put any number into our inverse trig functions. There are restrictions on the domain that are given in the following table:

	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

NOTE: in some textbooks, the inverse functions are written differently, for example instead of $y = \sin^{-1} x$, some textbooks may write this as $y = \arcsin x$. So instead of the $^{-1}$ symbol, it is replaced by the word *arc*. These two mean exactly the same thing. So $y = \arccos x$ would mean the same as $y = \cos^{-1} x$, etc.

EXAMPLE: Find the $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{2}$." If you look on your table of values, go to the sine column and go down until you see the value $\frac{\sqrt{3}}{2}$. This corresponds to an angle of 60 degrees, which is the answer.

EXAMPLE: Find the $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

What this is really asking is: "find an angle between 0 and π that has a value of $\frac{\sqrt{2}}{2}$." If you look on your table of values, go to the cosine column and go down until you see the value $\frac{\sqrt{2}}{2}$. This also corresponds to an angle of 45 degrees, which is the answer.

EXAMPLE: Find the $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{3}$." If you look on your table of values, go to the tangent column and go down until you see the value $\frac{\sqrt{3}}{3}$. This also corresponds to an angle of 30 degrees, which is the answer.

EXAMPLE: Use a calculator to find $\cos^{-1} 0.7$, if possible, where $0 < \theta < 2\pi$. Round your answer to two decimal places.

We need to make sure our calculator is in radian mode before we proceed. The inverse cosine is above the cosine key on your calculator. You will probably need to use your second key in order to get the inverse cosine. Your answer should be 0.8. If you got an error, try entering 0.7 first and then get the inverse.

EXAMPLE: Use a calculator to find $\sin^{-1}(-1.2)$, if possible, where $0 < \theta < 2\pi$. Round your answer to two decimal places.

If you try putting this in your calculator you will get an error. This is because 1.2 is not in our domain. Recall that the domain for the inverse sine function is $-1 \le x \le 1$. This means we can only put in numbers between -1 and 1. So the answer is no solution.

Inverses and canceling

If we take $\cos^{-1}(\cos x)$ what will we get? Well, the inverse cosine and cosine will cancel and that will leave us with just x. However there are some restrictions on what x can be as listed below:

$$\cos^{-1}(\cos x) = x \quad \textit{if} \quad 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \quad \textit{if} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \textit{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \textit{if} \quad -1 \le x \le 1$$
$$\tan^{-1}(\tan x) = x \quad \textit{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\tan(\tan^{-1} x) = x \quad \textit{if} \quad -\infty < x < \infty$$

EXAMPLE: Find the exact value if possible: $\tan(\tan^{-1} 5.3)$.

According to our restrictions above, x can be any number, so $\tan(\tan^{-1} 5.3) = 5.3$.

EXAMPLE: Find the exact value if possible: $\sin\left(\sin^{-1}\frac{98}{99}\right)$.

The fraction 98/99 is .9898, and this is less than 1, so $\sin\left(\sin^{-1}\frac{98}{99}\right) = \frac{98}{99}$.

EXAMPLE: Find the exact value if possible: $\cos\left(\cos^{-1}\sqrt{2}\right)$.

The square root changed into a decimal is 1.41, which is bigger than 1, so the answer is no solution since 1.41 is not in our domain.

EXAMPLE: Find the exact value: $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$

Since $\frac{\pi}{3}$ is in the domain $0 \le y \le \pi$, then our properties tell us $\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$.

EXAMPLE: Find the exact value: $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

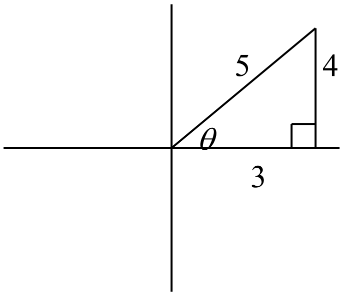
Be careful on this one. The answer is not $\frac{2\pi}{3}$. This is because $\frac{2\pi}{3}$ is not in $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. So we need to evaluate inside the parenthesis first, and then take the inverse. So we can use either a unit circle or reference angles to find $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. So now our problem becomes $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. We want to find an angle in the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ that has a y value of $\frac{\sqrt{3}}{2}$. Since sine is positive in the first quadrant, our answer is $\frac{\pi}{3}$. So $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

EXAMPLE: Find the exact value: $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$.

The answer is not $\frac{5\pi}{6}$. This is because $\frac{5\pi}{6}$ is not in $-\frac{\pi}{2} < y < \frac{\pi}{2}$. So we need to evaluate inside the parenthesis first, and then take the inverse. So we can use either a unit circle or reference angles to find $\tan\frac{5\pi}{6} = -\sqrt{3}$. So now our problem becomes $\tan^{-1}\left(-\sqrt{3}\right)$. We want to find an angle in the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ that has a y value of $-\sqrt{3}$. Since tangent is negative in the fourth quadrant, our answer is $-\frac{\pi}{6}$. So $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = -\frac{\pi}{6}$.

EXAMPLE: Use a sketch to find the exact value: $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$.

These problems involve drawing a triangle and labeling the sides like we did in a previous section. The inverse trig function will tell you where to draw the triangle. In our example there is an inverse cosine. The inverse cosine's range will tell us where we can draw the triangle. From the last section, the range for the inverse cosine is $0 \le y \le \pi$. This corresponds to the first and second quadrant. Since the fraction $\frac{3}{5}$ is positive, the only quadrant the triangle can be drawn in is the first quadrant. We know that the adjacent side is 3 and the hypotenuse is 5. The Pythagorean Theorem will give us the opposite side, which is 4.

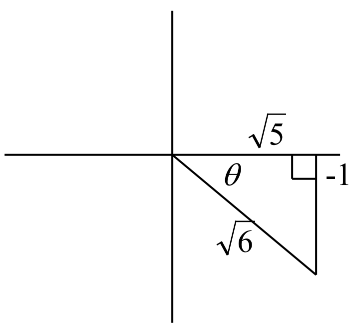


The sine on the outside of our problem tells us how to write our answer. From our drawing, sine is 4 over 5, so we write our answer as:

$$\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)=\frac{4}{5}$$

EXAMPLE: Use a sketch to find the exact value: $\cos\left(\sin^{-1}\left(-\frac{1}{\sqrt{6}}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in this case we have an inverse sine. The inverse sine's range will tell us where we can draw the triangle. From the last section, the range for the inverse sine is $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$. This corresponds to the first and fourth quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the fourth quadrant. We know that the opposite side is -1 and the hypotenuse is $\sqrt{6}$. The Pythagorean Theorem will give us the adjacent side, which is $\sqrt{5}$.

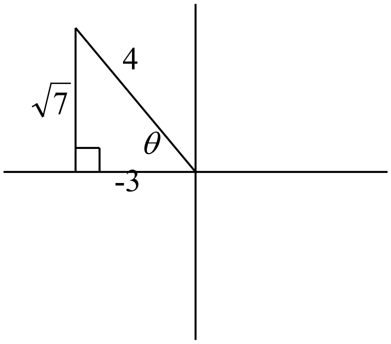


The cosine on the outside of our problem tells us how to write our answer. From our drawing, cosine is $\sqrt{5}$ over $\sqrt{6}$, so we write our answer as:

$$\cos\left(\sin^{-1}\left(-\frac{1}{\sqrt{6}}\right)\right)=\frac{\sqrt{5}}{\sqrt{6}}=\frac{\sqrt{30}}{6}$$

EXAMPLE: Use a sketch to find the exact value: $\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in our case there is an inverse cosine. The inverse cosine's range will tell us where we can draw the triangle. From the last section, the range for the inverse cosine is $0\le y\le\pi$. This corresponds to the first and second quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the second quadrant. We know that the adjacent side is -3 and the hypotenuse is 4. The Pythagorean Theorem will give us the opposite side: $\sqrt{7}$.

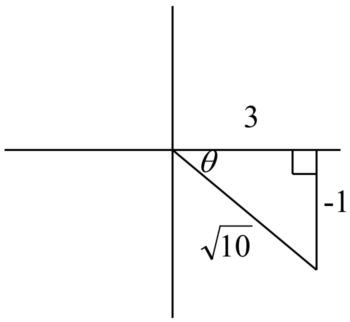


The tangent on the outside of our problem tells us how to write our answer. From our drawing, tangent is $\sqrt{7}$ over -3, so:

$$\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right)=-\frac{\sqrt{7}}{3}$$

EXAMPLE: Find the exact value: $\csc\left(\tan^{-1}\left(-\frac{1}{3}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in this case we have an inverse tangent. The inverse tangent's range will tell us where we can draw the triangle. From the last section, the range for the inverse tangent is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. This corresponds to the first and fourth quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the fourth quadrant. We know that the opposite side is -1 and the adjacent is 3. The Pythagorean Theorem will give us the adjacent side: $\sqrt{10}$.

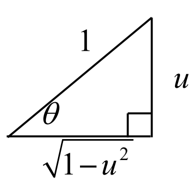


The cosecant on the outside of our problem tells us how to write our answer. From our drawing, cosecant is $\sqrt{10}$ over -1, so we write your answer as:

$$\csc\left(\tan^{-1}\left(-\frac{1}{3}\right)\right) = \frac{\sqrt{10}}{-1} = -\sqrt{10}\,.$$

EXAMPLE: Use right triangles to write in algebraic form: $\cos\left(\sin^{-1} u\right)$. Assume that *u* is positive and that the given inverse trigonometric function is defined for the expression in *u*.

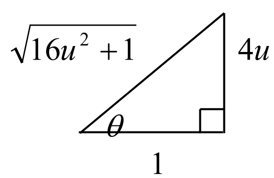
These problems involve drawing a triangle and labeling the sides with algebraic expressions. In this problem we are told that *u* is positive, so the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\cos\left(\sin^{-1}\frac{u}{1}\right)$ We know that the opposite side is *u* and the hypotenuse is 1. We can use the Pythagorean theorem to find the hypotenuse: $1^2 = u^2 + b^2 \;\Rightarrow\; b^2 = 1 - u^2$ So we have $b = \sqrt{1-u^2}$



The cosine on the outside of our problem tells us how to write our answer. From our drawing, cosine is $\sqrt{1-u^2}$ over 1 so we write our answer as: $\cos\left(\sin^{-1} u\right) = \sqrt{1-u^2}$.

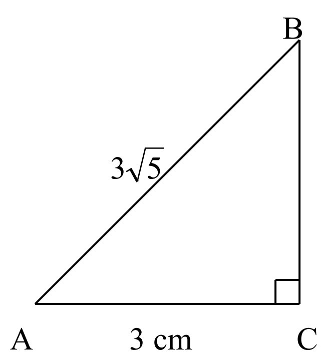
EXAMPLE: Use right triangles to write in algebraic form: $\sec\left(\tan^{-1} 4u\right)$. Assume that u is positive and that the given inverse trigonometric function is defined for the expression in u.

Again we will draw a triangle for this one. In this problem we are told that u is positive, so the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\tan^{-1}\frac{4u}{1}\right)$ We know that the adjacent side is 1 and the opposite side is $4u$. We can use the Pythagorean theorem to find the hypotenuse: $c^2 = (4u)^2 + (1)^2$. So we have $c = \sqrt{16u^2 + 1}$



The secant on the outside of our problem tells us how to write our answer. From our drawing, secant is $\sqrt{16u^2 + 1}$ over 1 so we write our answer as: $\sec\left(\tan^{-1} 4u\right) = \sqrt{16u^2 + 1}$

EXAMPLE: Solve the triangle:



I will let side BC be x. Now we can use the Pythagorean Theorem to find find it: $3^2 + x^2 = \left(3\sqrt{5}\right)^2$. Solving this will give us: $9 + x^2 = 45$, so $x = 6$. To find $m\angle A$, we can set up the following trig equation: $\cos A = \frac{3}{3\sqrt{5}}$. So we have $\cos A = 0.4472$. We need to take the inverse cosine to get our answer. So $A = \cos^{-1} 0.4472 = 63.44^\circ$. To find $m\angle B$ we will subtract 90 degrees and 63.44 degrees from 180 degrees. We will get: $m\angle B = 180 - 63.44 - 90 = 26.56^\circ$. Now our triangle is solved.