

## 5.2 Sum and Difference Identities

These formulas will allow use to find the exact value for other angles besides just 0, 30, 45, 60, and 90 degrees. The book uses the symbols  $\alpha$  and  $\beta$  which are more confusing. Instead I will use x and y.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

EXAMPLE: Determine the exact value of  $\sin 75^\circ$ .

We don't have 75 degrees on our table. We need to rewrite 75 degrees in terms of angle we have on our table. One way is to rewrite this as  $\sin(30^\circ + 45^\circ)$ . Then we can use the formula  $\sin(x + y) = \sin x \cos y + \cos x \sin y$  where x is 30 and y is 45:  $\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$ . Now everything here we can get

values from our table to get the exact answer:  $\sin(30^\circ + 45^\circ) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$ . After multiplying this we

$$\text{will get: } \sin(30^\circ + 45^\circ) = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

EXAMPLE: Determine the exact value of  $\cos 165^\circ$ .

We need to rewrite 165 degrees in terms of angle we have on our table. One way is to rewrite this as  $\cos(45^\circ + 120^\circ)$ . Then we can use the formula  $\cos(x + y) = \cos x \cos y - \sin x \sin y$  where x is 45 and y is 120:  $\cos(45^\circ + 120^\circ) = \cos 45^\circ \cos 120^\circ + \sin 45^\circ \sin 120^\circ$ . Now everything here we can get values from our table to get the exact answer. At 120 degrees the reference angle is 60 degrees. Since 120 degree is in the second

quadrant, the cosine is negative and the sine is positive. So  $\cos 120^\circ = -\frac{1}{2}$  and  $\sin 120^\circ = \frac{\sqrt{3}}{2}$ . So

$$\cos(45^\circ + 120^\circ) = \frac{\sqrt{2}}{2} \cdot -\frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}. \text{ After multiplying this we will get: } \sin(30^\circ + 45^\circ) = \frac{-\sqrt{2} - \sqrt{6}}{4}.$$

EXAMPLE: Determine the exact value:  $\tan 105^\circ$ .

We need to rewrite 105 degrees as:  $\tan(45^\circ + 60^\circ)$ . Using the fifth formula we get:

$$\tan(45^\circ + 60^\circ) = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}. \text{ Using values from our table we will get:}$$

$$\tan(45^\circ + 60^\circ) = \frac{1 + \sqrt{3}}{1 - (1)\sqrt{3}}. \quad \text{We need to rationalize the denominator.}$$

$$\tan(45^\circ + 60^\circ) = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \quad \text{Now multiply and simplify.}$$

$$\tan(45^\circ + 60^\circ) = \frac{1 + 2\sqrt{3} + 3}{1 - 3} \quad \text{Simplify the top.}$$

$$\tan(45^\circ + 60^\circ) = \frac{4 + 2\sqrt{3}}{-2} \quad \text{Now reduce.}$$

$$\tan(105^\circ) = -2 - \sqrt{3}$$

EXAMPLE: Write as a single trig value and then find the exact value:  $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$ .

We have sine cosine and then cosine sine. This is using the formula  $\sin(x + y) = \sin x \cos y + \cos x \sin y$ . We know that  $x = 40^\circ$  and  $y = 20^\circ$ . This means that the whole expression above turns into  $\sin(40^\circ + 20^\circ)$ . This simplifies to  $\sin(60^\circ)$ . From our table we know that  $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ .

EXAMPLE: Write as a single trig value and then find the exact value:  $\cos\left(\frac{3\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) - \sin\left(\frac{3\pi}{10}\right)\sin\left(\frac{\pi}{5}\right)$ .

We have cosine cosine and then sine sine. This is using the formula  $\cos(x + y) = \cos x \cos y - \sin x \sin y$ . We know that  $x = \frac{3\pi}{10}$  and  $y = \frac{\pi}{5}$ . This means that the whole expression above turns into  $\cos\left(\frac{3\pi}{10} + \frac{\pi}{5}\right)$ . This simplifies to  $\cos\left(\frac{\pi}{2}\right)$ . From our table we know that  $\cos\left(\frac{\pi}{2}\right) = 0$ .

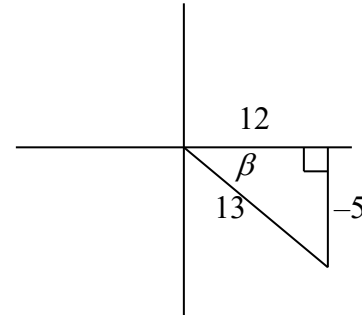
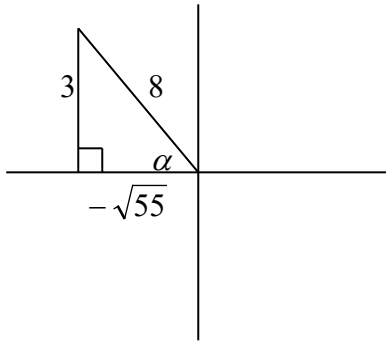
EXAMPLE: Write as a single trig value and then find the exact value:  $\frac{\tan 50^\circ - \tan 20^\circ}{1 + \tan 50^\circ \tan 20^\circ}$ .

This one is written in the form  $\frac{\tan x - \tan y}{1 + \tan x \tan y}$ , which is the expression for  $\tan(x - y)$ . Here, we will let

$x = 50^\circ$  and  $y = 20^\circ$ . So then the above fraction expression will turn into  $\tan(50^\circ - 20^\circ)$ . This simplifies into  $\tan(30^\circ)$ . The table of trig values tells us that this is equal to  $\frac{\sqrt{3}}{3}$ .

EXAMPLE: Find the exact value of  $\sin(\alpha + \beta)$  given  $\sin(\alpha) = \frac{3}{8}$  for  $\alpha$  in Quadrant II and  $\cos(\beta) = \frac{12}{13}$  for  $\beta$  in Quadrant IV.

Using the difference formula we know  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ . We have some of these values given to us, however we need to find  $\cos \alpha$  and  $\sin \beta$ . To do this we need to make triangles and use the Pythagorean theorem. They tell us what quadrant each is in, so we can now draw triangles and solve for the missing sides:



From our triangles we see that  $\cos \alpha = -\frac{\sqrt{55}}{8}$  and  $\sin \beta = -\frac{5}{13}$ . We can now plug all our values back into the

difference formula:  $\sin(\alpha + \beta) = \frac{3}{8} \cdot \frac{12}{13} + \left(-\frac{\sqrt{55}}{8}\right) \left(-\frac{5}{13}\right)$ . This can be simplified to  $\frac{36 + 5\sqrt{55}}{104}$ .

Now we will look at solving some proofs, which was done in the previous section.

EXAMPLE: Establish the identity:  $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos \theta$ .

We will start by working on the left side of the identity. We will use the second formula above, which is

$\sin(x - y) = \sin x \cos y - \cos x \sin y$ . Here the  $x$  would be  $\frac{3\pi}{2}$  and the  $y$  is  $\theta$ :

$$\sin\left(\frac{3\pi}{2} - \theta\right) = \sin\left(\frac{3\pi}{2}\right) \cos \theta - \cos\left(\frac{3\pi}{2}\right) \sin \theta \quad \text{We know } \sin\left(\frac{3\pi}{2}\right) = -1 \quad \text{and} \quad \cos\left(\frac{3\pi}{2}\right) = 0. \text{ Substitute.}$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -1 \cdot \cos \theta - 0 \cdot \sin \theta \quad \text{This simplifies to } -\cos \theta.$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos \theta.$$

EXAMPLE: Establish the identity:  $\frac{\cos(x + 3\pi/2)}{\sin(x - \pi/2)} = -\tan x$ .

For this one we will first use the sum and difference formulas on the top and bottom:

$$\frac{\cos x \cos(3\pi/2) - \sin x \sin(3\pi/2)}{\sin x \cos(\pi/2) - \cos x \sin(\pi/2)} = -\tan x \quad \frac{3\pi}{2} \text{ is } 270 \text{ degrees, which has a reference angle of } 90^\circ.$$

$$\frac{\cos x \cdot 0 - \sin x \cdot -1}{\sin x \cdot 0 - \cos x \cdot 1} = -\tan x$$

Now simplify.

$$-\frac{\sin x}{\cos x} = -\tan x$$

$$-\tan x = -\tan x$$

Both sides are equal, so we are done.

EXAMPLE: Establish the identity:  $\frac{\sin(x + y)}{\cos x \cos y} = \tan x + \tan y$ .

We need to use the formula  $\sin(x + y) = \sin x \cos y + \cos x \sin y$ .

$$\frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y} = \tan x + \tan y$$

Now divide each term on top by what is on the bottom.

$$\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y} = \tan x + \tan y$$

We can cancel.

$$\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \tan x + \tan y$$

Sine over cosine is tangent.

$$\tan x + \tan y = \tan x + \tan y$$

Both sides are equal, so we are done.

One more...

EXAMPLE: Establish the identity:  $\frac{\cos(x+y)}{\cos(x-y)} = \frac{1 - \tan x \tan y}{1 + \tan x \tan y}$ .

Start by changing the tangent to sines and cosines.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{1 - \frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y}}{1 + \frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y}}$$

Now get common denominators.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}$$

Combine fractions together.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\frac{\cos x \sin y - \sin x \sin y}{\cos x \sin y}}{\frac{\cos x \sin y + \sin x \sin y}{\cos x \sin y}}$$

Flip the bottom fraction and multiply.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x \sin y - \sin x \sin y}{\cos x \sin y} \cdot \frac{\cos x \sin y}{\cos x \sin y + \sin x \sin y}$$

Cancel the  $\cos x \sin y$ .

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x \sin y - \sin x \sin y}{\cos x \sin y + \sin x \sin y}$$

Use the sum and difference formulas.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos(x+y)}{\cos(x-y)}$$

Both sides are equal so we are done.