

5.3 Double Angle, Half Angle, and Reduction Formulas

If we have either a double angle 2θ or a half angle $\frac{\theta}{2}$ then these have special formulas:

Double Angle Formulas

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos(2\theta) = 2 \cos^2 \theta - 1 \quad \text{There are three formulas for } \cos(2\theta)$$

$$\cos(2\theta) = 1 - 2 \sin^2 \theta$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

The above double angle formulas can be manipulated to derive power reducing formulas. These formulas will be useful primarily in Calculus 2:

Power Reducing Formulas

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \quad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \quad \tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Besides double angle formulas there are also half angle formulas:

Half Angle Formulas

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$

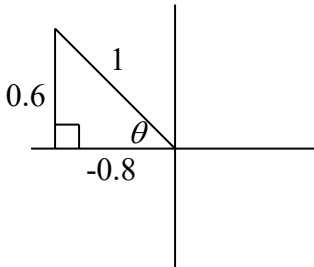
$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$

There are better formulas for $\tan \frac{\theta}{2}$ that don't involve a plus or minus:

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}, \quad \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$$

EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ if you are given $\cos \theta = -0.8$ and $90^\circ \leq \theta \leq 180^\circ$. Round decimals to two decimal places.

We need to draw a triangle for this one. We are given that the triangle should be drawn in the second quadrant. We can rewrite our problem as $\cos \theta = \frac{-0.8}{1}$. We know the adjacent side is -0.8 . The hypotenuse is 1. Once we label our triangle we find the third side by using the Pythagorean theorem.



Now we can get our first 5 trigonometric functions by reading off our triangle:

$$\sin \theta = 0.6 \qquad \csc \theta = \frac{1}{0.6} = 1.67 \qquad \sec \theta = \frac{1}{-0.8} = -1.25$$

$$\tan \theta = \frac{0.6}{-0.8} = -0.75 \qquad \cot \theta = \frac{-0.8}{0.6} = -1.33$$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2 \sin \theta \cos \theta$. We already know sine and cosine, so we will substitute in those decimals: $\sin(2\theta) = 2(0.6)(-0.8) = -0.96$.

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2 \cos^2 \theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos \theta = -0.8$, so we will substitute in this decimal: $\cos(2\theta) = 2(-0.8)^2 - 1 = 0.28$.

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$. We already know $\tan \theta = -0.75$, so we

$$\text{will substitute in this decimal: } \tan(2\theta) = \frac{2(-0.75)}{1 - (-0.75)^2} = \frac{-1.5}{0.4375} = -3.43.$$

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let's first start with our given statement $90^\circ \leq \theta \leq 180^\circ$. If I divide everything by two we get: $45^\circ \leq \frac{\theta}{2} \leq 90^\circ$. This tells us that $\frac{\theta}{2}$ is in the first quadrant, so sine, cosine, and tangent of $\frac{\theta}{2}$ should all be positive.

Now I will find $\sin \frac{\theta}{2}$ by using its formula: $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the first quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - (-0.8)}{2}} = \sqrt{\frac{1.8}{2}} = \sqrt{0.9} = 0.95.$$

Now I will find $\cos \frac{\theta}{2}$ by using its formula: $\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the first quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + (-0.8)}{2}} = \sqrt{\frac{0.2}{2}} = \sqrt{0.1} = 0.32.$$

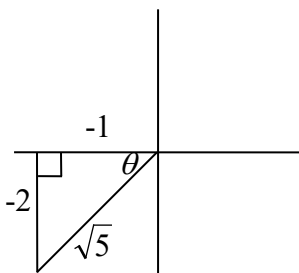
Finally I will find $\tan \frac{\theta}{2}$. I have three formulas to choose. I will choose: $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$. We already know

$$\cos \theta = -0.8, \text{ and } \sin \theta = 0.6 \text{ so we will substitute in these decimals: } \tan \frac{\theta}{2} = \frac{0.6}{1 + (-0.8)} = 3.$$

EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$

if you are given $\cot \theta = \frac{1}{2}$ and $\sin \theta < 0$.

We are given that the cotangent is positive and sine is negative. This only occurs in the 3rd quadrant. We know the adjacent side is 1 and the opposite side is 2. However, because we are in the third quadrant we need to make the 1 and 2 negative. Once we label our triangle we find the third side by using the Pythagorean theorem. This will give us $\sqrt{5}$.



Now we can get our first 5 trigonometric functions by reading off our triangle:

$$\sin \theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \qquad \cos \theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5} \qquad \sec \theta = -\sqrt{5}$$

$$\csc \theta = -\frac{\sqrt{5}}{2} \qquad \tan \theta = 2$$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2\sin\theta\cos\theta$. We already know sine and cosine, so we

will substitute in those fractions:
$$\sin(2\theta) = 2\left(-\frac{2\sqrt{5}}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) = \frac{20}{25} = \frac{4}{5}.$$

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2\cos^2\theta - 1$. Notice I had a choice of three formulas

to use. Any of them would give you the correct answer. We already know $\cos\theta = -\frac{\sqrt{5}}{5}$, so we will substitute

in this fraction:
$$\cos(2\theta) = 2\left(-\frac{\sqrt{5}}{5}\right)^2 - 1. \text{ This simplifies to: } 2\left(\frac{5}{25}\right) - 1 = \frac{2}{5} - 1 = -\frac{3}{5}.$$

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$. We already know $\tan\theta = 2$, so we will

substitute in this number:
$$\tan(2\theta) = \frac{2(2)}{1-(2)^2} = -\frac{4}{3}.$$

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let's first start with our

given statement $180^\circ \leq \theta \leq 270^\circ$. If I divide everything by two we get: $90^\circ \leq \frac{\theta}{2} \leq 135^\circ$. This tells us that $\frac{\theta}{2}$

is in the second quadrant, so sine of $\frac{\theta}{2}$ should be positive, and the cosine and tangent of $\frac{\theta}{2}$ should be negative.

Now I will find $\sin\frac{\theta}{2}$ by using its formula: $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos\theta = -0.8$, so we will substitute in this decimal:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\left(-\frac{\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{5+\sqrt{5}}{5}} = \sqrt{\frac{5+\sqrt{5}}{10}}.$$

Now I will find $\cos\frac{\theta}{2}$ by using its formula: $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$. We chose a negative because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos\theta = -0.8$, so we will substitute in this decimal:

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\left(-\frac{\sqrt{5}}{5}\right)}{2}} = -\sqrt{\frac{5-\sqrt{5}}{5}} = -\sqrt{\frac{5-\sqrt{5}}{10}}.$$

Finally I will find $\tan \frac{\theta}{2}$. I have three formulas to choose. I will choose: $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$. We already know $\cos \theta = -0.8$, and $\sin \theta = 0.6$ so we will substitute in these decimals:

$$\tan \frac{\theta}{2} = \frac{-\frac{2\sqrt{5}}{5}}{1 + \left(-\frac{\sqrt{5}}{5}\right)} = \frac{-\frac{2\sqrt{5}}{5}}{\frac{5 - \sqrt{5}}{5}} = -\frac{2\sqrt{5}}{5} \cdot \frac{5}{5 - \sqrt{5}} = -\frac{2\sqrt{5}}{5 - \sqrt{5}} \cdot \frac{5 + \sqrt{5}}{5 + \sqrt{5}} = \frac{-10\sqrt{5} - 10}{25 - 5} = \frac{-10\sqrt{5} - 10}{20} = \frac{-\sqrt{5} - 1}{2}.$$

EXAMPLE: Compute $\sin(22.5^\circ)$ and $\tan(22.5^\circ)$ using a half-angle formula.

We can write this as $\sin\left(\frac{45^\circ}{2}\right)$. Then we know that θ is 45 degrees, so now we use: $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos 45^\circ}{2}}$. It

is positive since 22.5 is in the first quadrant. You get: $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$.

For $\tan(22.5^\circ)$ we will use: $\tan \frac{45^\circ}{2} = \frac{\sin 45^\circ}{1 + \cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{2 + \sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1$.

EXAMPLE: Rewrite $1 - 2\sin^2(\pi/12)$ using a double angle formula. Then find its exact value.

This one is written in the form $1 - 2\sin^2 \theta$, which is a form of $\cos 2\theta$. In this problem, we will let $\theta = \pi/12$.

Therefore we will substitute this into the formula $\cos 2\theta$: $\cos\left[2\left(\frac{\pi}{12}\right)\right]$. Simplifying gives us $\cos \frac{\pi}{6}$. Looking

at our table we see this has an exact value of $\sqrt{3}/2$.

EXAMPLE: Establish the identity: $\frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = \cos 2\theta$

First we want to change these into sines and cosines.

$$\frac{\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta}} = \cos 2\theta$$

Now get common denominators on the top and bottom.

$$\frac{\left(\frac{\cos \theta}{\cos \theta}\right) \cdot \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \cdot \left(\frac{\sin \theta}{\sin \theta}\right)}{\left(\frac{\cos \theta}{\cos \theta}\right) \cdot \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \cdot \left(\frac{\sin \theta}{\sin \theta}\right)} = \cos 2\theta \quad \text{Multiply and write over a single denominator}$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\frac{\cos \theta \sin \theta}{\cos^2 \theta + \sin^2 \theta}} = \cos 2\theta$$

$$\frac{\cos \theta \sin \theta}{\cos \theta \sin \theta}$$

We will get rid of the double fractions and use $\cos^2 \theta + \sin^2 \theta = 1$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

The left side is the identity for $\cos 2\theta$.

$$\cos 2\theta = \cos 2\theta$$

EXAMPLE: $(4 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta) = \sin 4\theta$

First we will use the identity $1 - 2 \sin^2 \theta = \cos 2\theta$. Now our problem becomes:

$$(4 \sin \theta \cos \theta) \cos 2\theta = \sin 4\theta$$

We know that $\sin 2\theta = 2 \sin \theta \cos \theta$. We want to rewrite our problem as the following:

$$2 \cdot 2 \sin \theta \cos \theta \cdot \cos 2\theta = \sin 4\theta \quad \text{Now we can use the identity } \sin 2\theta = 2 \sin \theta \cos \theta.$$

$$2 \sin 2\theta \cos 2\theta = \sin 4\theta \quad \text{We know that } 2 \sin 2\theta \cos 2\theta \text{ is the same as } \sin(2 \cdot 2\theta) = \sin 4\theta.$$

$$\sin 4\theta = \sin 4\theta$$

EXAMPLE: Use power-reducing formulas to rewrite $\cos^4 \theta$ in terms of first powers of cosine.

First we can rewrite the original problem as: $\cos^2 \theta \cdot \cos^2 \theta$. This will allow us to use the power reducing formula for cosine. We will use the same formula twice: $\frac{1 + \cos(2\theta)}{2} \cdot \frac{1 + \cos(2\theta)}{2}$. Now multiply across the

top and bottom to get $\frac{1 + 2 \cos(2\theta) + \cos^2(2\theta)}{4}$. We can split up the fraction: $\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos^2(2\theta)$.

Now we will use another power reducing formula: $\cos^2(2\theta) = \frac{1 + \cos(4\theta)}{2}$. When you substitute this in you

will get $\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cdot \frac{1 + \cos(4\theta)}{2}$. Multiply to get $\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1 + \cos(4\theta)}{8}$. Now split up the last

fraction: $\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} + \frac{\cos(4\theta)}{8}$. Add the fractions to get the final answer: $\frac{3}{8} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta)$