5.4 Sum-to-Product and Product-to-Sum Formulas

Product-to-Sum Formulas

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$
$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

EXAMPLE: Simplify: $\sin(6\theta)\sin(4\theta)$ using a product-to-sum formula.

We will use the first formula $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$ with $x = 6\theta$ and $y = 4\theta$

$\frac{1}{2}[\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)]$ Now simplify.

$\frac{1}{2}[\cos(2\theta) - \cos(10\theta)]$ This is as far as we can go.

EXAMPLE: Simplify: $\cos(3\theta)\cos(\theta)$ using a product-to-sum formula.

We will use the formula $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$ with $x = 3\theta$ and $y = \theta$.

$\frac{1}{2}[\cos(3\theta - \theta) + \cos(3\theta + \theta)]$ Now simplify.

$\frac{1}{2}[\cos(2\theta) + \cos(4\theta)]$ This is as far as we can go.

EXAMPLE: Simplify: $\sin(3\theta)\cos(5\theta)$ using a product-to-sum formula.

We will use the formula $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$ with $x = 3\theta$ and $y = 5\theta$.

$\frac{1}{2}[\sin(3\theta - 5\theta) + \sin(3\theta + 5\theta)]$ Now simplify.

$\frac{1}{2}[\sin(-2\theta) + \sin(8\theta)]$ We will use the identity $\sin(-2\theta) = \sin 2\theta$.

$\frac{1}{2}[-\sin(2\theta) + \sin(8\theta)]$ or $\frac{1}{2}[\sin(8\theta) - \sin(2\theta)]$ is as far as we can go.

EXAMPLE: Find the exact value of $\cos\frac{5\pi}{12}\sin\frac{\pi}{12}$ using a product-to-sum formula.

We will use the formula $\cos x \sin y = \frac{1}{2}[\sin(x+y)-\sin(x-y)]$ with $x=\frac{5\pi}{12}$ and $y=\frac{\pi}{12}$.

$\frac{1}{2}\left[\sin\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)-\sin\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)\right]$ Now simplify.

$\frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{3}\right)\right]$ We can use our table to get the values of these trig functions.

$\frac{1}{2}\left[1-\frac{\sqrt{3}}{2}\right]=\frac{1}{2}-\frac{\sqrt{3}}{4}=\frac{2-\sqrt{3}}{4}$.

Sum-to-Product Formulas

$$\sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x-\sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos x+\cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x-\cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

EXAMPLE: Simplify: $\sin(5\theta)-\sin(3\theta)$ using a sum-to-product formula.

We will use the formula $\sin x-\sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$ with $x=5\theta$ and $y=3\theta$.

$2\sin\left(\frac{5\theta-3\theta}{2}\right)\cos\left(\frac{5\theta+3\theta}{2}\right)$ Now simplify.

$2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{8\theta}{2}\right)$

$2\sin(\theta)\cos(4\theta)$ We can’t simplify this anymore, so we are done.

EXAMPLE: Simplify: $\cos(3\theta)+\cos(2\theta)$ using a sum-to-product formula.

We will use the formula $\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ with $x=3\theta$ and $y=2\theta$.

$2\cos\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right)$ Now simplify.

$2\cos\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ We can't simplify anymore, so we are done.

EXAMPLE: Simplify: $\cos(4\theta)-\cos(7\theta)$ using a sum-to-product formula.

We will use the formula $\cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ with $x=4\theta$ and $y=7\theta$.

$-2\sin\left(\frac{4\theta+7\theta}{2}\right)\sin\left(\frac{4\theta-7\theta}{2}\right)$ Now simplify.

$-2\sin\left(\frac{11\theta}{2}\right)\sin\left(\frac{-3\theta}{2}\right)$ We can use the identity $\sin\left(\frac{-3\theta}{2}\right)=-\sin\left(\frac{3\theta}{2}\right)$.

$-2\sin\left(\frac{11\theta}{2}\right)\cdot-\sin\left(\frac{3\theta}{2}\right)$

$2\sin\left(\frac{11\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$

EXAMPLE: Find the exact value of $\sin 15^\circ+\sin 75^\circ$ using a sum-to-product formula.

We will use the formula $\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ with $x=15^\circ$ and $y=75^\circ$.

$2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\cos\left(\frac{15^\circ-75^\circ}{2}\right)$ Simplify.

$2\sin(45^\circ)\cos(-30^\circ)$ From here we can use our table to get the exact values.

$2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{6}}{2}$.