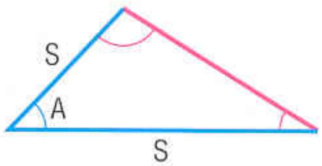
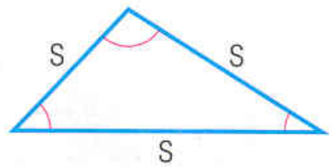
6.2 Non-right Triangles: Law of Cosines

The law of cosines is used to solve for missing sides or angles of triangles when we have the following two cases:



SAS – Side Angle Side

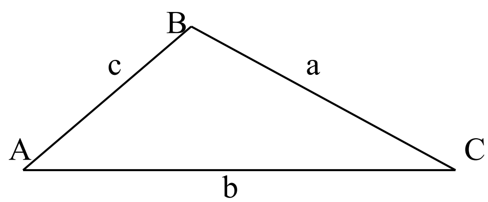


SSS – Side Side Side

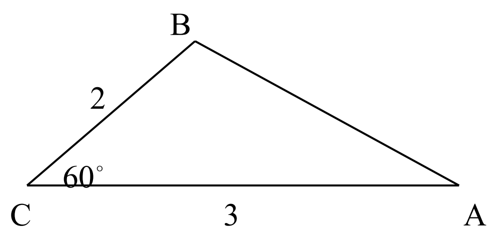
The law of cosines can be written three different ways depending on what you are trying to solve for.

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$



EXAMPLE: Solve the triangle:



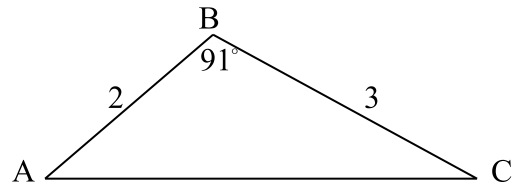
The only side we can find right now is side c since this is opposite angle C which is given. We need to use this version of the Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$. Here a = 2, b = 3, and C = 60°. So we have: $c^2 = 2^2 + 3^2 - 2(2)(3)\cos 60^\circ$. Simplifying we get: $c^2 = 4 + 9 - 12(0.5)$. So $c^2 = 7$ and so $c = \sqrt{7}$.

In order to solve for angle A we can either use the law of cosines again or we can use the law of sines. I will use the law of sines: $\frac{\sin 60^\circ}{\sqrt{7}} = \frac{\sin A}{2}$. Cross multiplying will give us: $2\sin 60^\circ = \sqrt{7}\sin A$. So

$\sin A = \frac{2\sin 60^\circ}{\sqrt{7}}$. We get $\sin A = 0.6547$. After taking the inverse sine we get $m\angle A \approx 40.89$. We also get a second answer for A: $m\angle A \approx 180 - 40.89 = 139.11$. This is not possible since there is already an angle of 60 degrees inside the triangle. Also if we already know three sides of the triangle there can only be one possible solution.

Lastly we will find angle B: $m\angle B = 180 - 60 - 40.89 = 79.1^\circ$.

EXAMPLE: Solve the triangle:

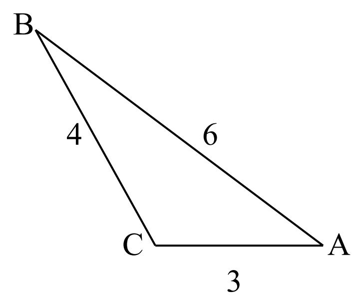


The only side we can find right now is side b since this is opposite angle B which is given. We need to use this version of the Law of Cosines: $b^2 = a^2 + c^2 - 2ac\cos B$. Here a = 3, b = 2, and C = 91°. So we have: $b^2 = 3^2 + 2^2 - 2(3)(2)\cos 91^\circ$. Simplifying we get: $b^2 = 9 + 4 - 12(-0.0175)$. So $b^2 = 13.21$ and so $b = 3.63$.

In order to solve for angle A we can either use the law of cosines again or we can use the law of sines. I will use the law of sines: $\frac{\sin 91^\circ}{3.63} = \frac{\sin A}{3}$. Cross multiplying will give us: $3\sin 91^\circ = 3.63\sin A$. So $\sin A = \frac{3\sin 91^\circ}{3.63}$. We get $\sin A = 0.8263$. After taking the inverse sine we get $m\angle A \approx 55.72$.

Lastly we will find angle B: $m\angle B = 180 - 91 - 55.72 = 33.28^\circ$.

EXAMPLE: Solve the triangle:



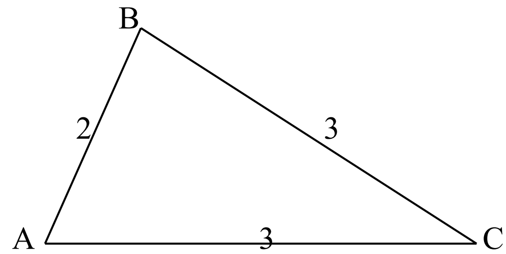
We can solve for whatever angle we want since all three sides are given. So I will just solve for angle A first. We need to use $a^2 = b^2 + c^2 - 2bc\cos A$. Here a = 4, b = 3, and c = 6. So we have $4^2 = 3^2 + 6^2 - 2(3)(6)\cos A$ which after simplification is: $16 = 9 + 36 - 36\cos A$. Then we have: $16 = 45 - 36\cos A$, and $-29 = -36\cos A$, so $\cos A = 0.8055$. The inverse cosine will give us $m\angle A = 36.34^\circ$. Will there be a second answer? The answer is no because cosine is only positive in the first and fourth quadrant. The fourth quadrant has angles between 270 and 360 degrees, which are impossible in a triangle, so we have only one solution.

Let's now solve for angle C. We can either use the law of cosines again or we can use the law of sines. I will use the law of sines: $\frac{\sin 36.34^\circ}{4} = \frac{\sin C}{6}$. Cross multiplying will give us: $6\sin 36.34^\circ = 4\sin C$. So $\sin C = \frac{6\sin 36.34^\circ}{4}$. We get $\sin C = 0.8889$. After taking the inverse sine we get $m\angle C \approx 62.73$. What is wrong with this? If we look at our picture it shows that angle C should be more than 90 degrees. We know there should only be one solution to this triangle so we know that the angle we found is not our solution, so $m\angle C \approx 180 - 62.73 = 117.27^\circ$.

In order to avoid this and get the direct answer it is best to use the law of cosines a second time whenever you have a triangle where three sides are given. You will get the answer right away. Let's use the law of cosines to get angle C. We will use: $6^2 = 4^2 + 3^2 - 2(4)(3)\cos C$. This simplifies to: $36 = 16 + 9 - 24\cos C$. Then we have $\cos C = -0.4583$. After taking the inverse cosine we get $m\angle C \approx 117.28^\circ$.

Finally we can find angle B: $m\angle B = 180 - 36.34 - 117.28 = 26.38^\circ$.

EXAMPLE: Solve the triangle:

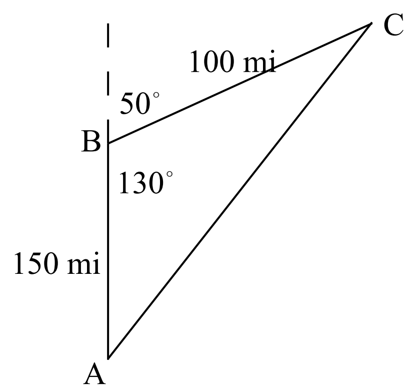


Since side BC and side AC are the same this means that angle A and angle B are equal. So we will find A first and then we know this will automatically be angle B. We will use $a^2 = b^2 + c^2 - 2bc\cos A$. In this problem we know a = 3, b = 3, and c = 2. Putting this into the formula we have: $3^2 = 3^2 + 2^2 - 2(3)(2)\cos A$. Then we have: $9 = 9 + 4 - 12\cos A$. Solving for cosine will give us $\cos A = 0.3333$. The inverse cosine will give us $m\angle A = 70.53^\circ$. So because we have an isosceles triangle we know that $m\angle B = 70.53^\circ$. Now we can find angle C: $m\angle C = 180 - 70.53 - 70.53 = 38.94^\circ$.

EXAMPLE: A plane flies due north from Ft. Myers to Sarasota, a distance of 150 miles. Then the plane flies at a bearing $N50^\circ E$ and flies to Orlando, a distance of 100 miles.

a.) How far is it from Ft. Myers to Orlando?

First we need to draw a picture. I will let point A = Ft. Myers B = Sarasota, and C = Orlando. Notice that the 50 degrees is measured from the north. Then we know that the angle B inside the triangle is $180^\circ - 50^\circ = 130^\circ$.



We need to find the distance from A to C, and we will call this b. We need to use this version of the Law of Cosines: $b^2 = a^2 + c^2 - 2ac\cos B$. We know a = 100, c = 150 and B = 50 degrees. Putting this into the formula we will have: $b^2 = 100^2 + 150^2 - 2(100)(150)\cos 130$. Then we have $b^2 = 10000 + 22500 - 30000(-0.6427)$. This give us $b^2 = 51783.63$. After taking the square root of both sides we get $b = 227.56$. So the distance from Ft. Myers to Orlando is approximately 227.56 miles.

b.) What bearing should the pilot use to fly directly from Ft. Myers to Orlando?

We need to find angle A since this is measured from the north. We can use the law of cosines again but this time we will use $a^2 = b^2 + c^2 - 2bc\cos A$. Now we know that a = 100, b = 227.56, and c = 150. Putting this into the formula will give us $100^2 = 227.56^2 + 150^2 - 2(227.56)(150)\cos A$. Simplifying will give us: $10000 = 51783.5536 + 22500 - 68268\cos A$. Combining like terms gives us: $10000 = 74283.5536 - 68268\cos A$. Then we have $-64283.5536 = -68268\cos A$. Solving for cosine we will get: $\cos A = 0.9416$. Then $m\angle A = 19.67^\circ$.

We need to write our answer as a bearing, so $N19.67^\circ E$ is the bearing the pilot must use to fly from Ft. Myers to Orlando.

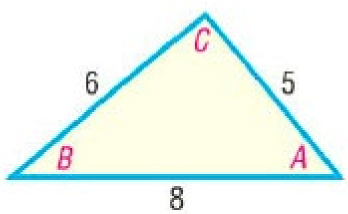
Heron's Formula

The area K of a triangle with sides a, b, and c is
$K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$

EXAMPLE: Find the area of a triangle given $a = 4$, $b = 5$, $c = 3$.

Because there is no angle given, we must use Heron's formula. First we need to calculate s using $s = \frac{1}{2}(a+b+c)$. So $s = \frac{1}{2}(4+5+3) = \frac{1}{2}(12) = 6$. So s = 6. Now we will use the formula $K = \sqrt{s(s-a)(s-b)(s-c)}$. Plug in the unknowns: $K = \sqrt{6(6-4)(6-5)(6-3)}$. Simplify: $K = \sqrt{6(2)(1)(3)} = \sqrt{36} = 6$. Therefore the area of this triangle is 6.

EXAMPLE: Find the area of the following triangle. Round your answer to two decimal places.



Because there is no angle given, we must use Heron's formula. From the triangle above we know $a = 6$, $b = 5$, and $c = 8$. First we need to calculate s using $s = \frac{1}{2}(a+b+c)$. So $s = \frac{1}{2}(6+5+8) = \frac{1}{2}(19) = 9.5$. So s = 9.5. Now we will use the formula $K = \sqrt{s(s-a)(s-b)(s-c)}$. Plug in the unknowns: $K = \sqrt{9.5(9.5-6)(9.5-5)(9.5-8)}$. Simplify: $K = \sqrt{9.5(3.5)(4.5)(1.5)} = \sqrt{224.4375} \approx 14.98$ rounded to two decimal places. Therefore the area of this triangle is 14.98.