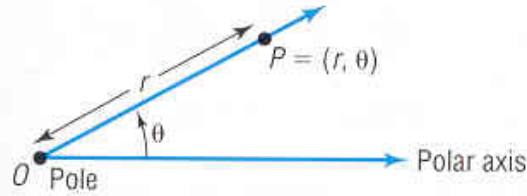


6.3 Polar Coordinates

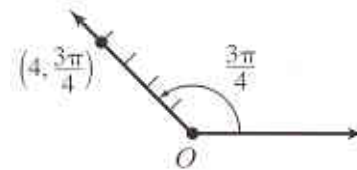
In this section we will learn a new coordinate system. In this system we plot a point in the form (r, θ) . As shown in the picture below you first draw angle θ in standard form. Then you label how long r is:



EXAMPLE: Plot $\left(4, \frac{3\pi}{4}\right)$ in the polar coordinate system.

We can convert $\frac{3\pi}{4}$ into degrees by multiplying by $\frac{180}{\pi}$.

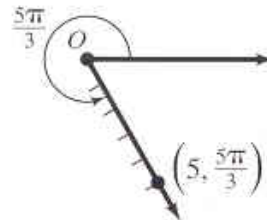
You will get 135 degrees. This is in the second quadrant. First draw the angle and then mark off 4 units to represent the radius.



EXAMPLE: Plot $\left(5, \frac{5\pi}{3}\right)$ in the polar coordinate system.

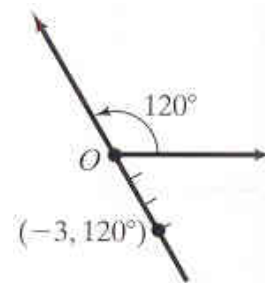
We can convert $\frac{5\pi}{3}$ into degrees by multiplying by $\frac{180}{\pi}$.

You will get 300 degrees. This is in the fourth quadrant. First draw the angle and then mark off 5 units to represent the radius.



EXAMPLE: Plot $(-3, 120^\circ)$ in the polar coordinate system.

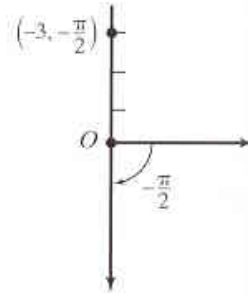
We already have it in degrees. This is in the second quadrant. First draw the angle in standard position. Since we have a negative radius, we must plot this differently. Instead of marking along our original angle, we will draw another angle exactly 180 degrees away from our original angle. Then we mark off 3 units.



EXAMPLE: Plot $\left(-3, -\frac{\pi}{2}\right)$ in the polar coordinate system.

We need to convert this into degrees by multiplying by $\frac{180}{\pi}$.

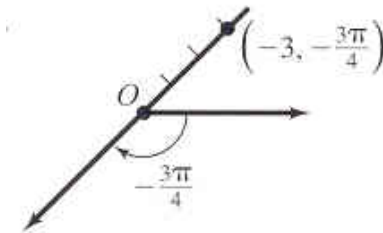
You will get -90 degrees. First draw the angle in standard position. Remember that negative angles are drawn clockwise in standard position. Since we have a negative radius instead of marking along our original angle, we will draw another angle exactly 180 degrees away from our original angle. Then we mark off 3 units.



EXAMPLE: Plot $\left(-3, -\frac{3\pi}{4}\right)$ in the polar coordinate system.

We need to convert this into degrees by multiplying by $\frac{180}{\pi}$.

You will get -135 degrees. First draw the angle in standard position. Remember that negative angles are drawn clockwise in standard position. Since we have a negative radius instead of marking along our original angle, we will draw another angle exactly 180 degrees away from our original angle. Then we mark off 3 units.



Equivalent Polar Coordinates

There are more than one way to arrive at the same angle. For example in the previous problem, -135 degrees is the same as $360^\circ + (-135^\circ) = 225^\circ$. If we have 120 degrees then this is the same as $120^\circ + (-360^\circ) = -240^\circ$. So for negative angles, just add 360 degrees. For positive angles add negative 360 degrees to find the equivalent angle. So basically we can either move clockwise or counterclockwise to arrive at the same angle.

$$\begin{aligned}(r, \theta) &= (r, \theta \pm 2\pi) \text{ or } (r, \theta) = (r, \theta \pm 360^\circ) \\ (r, \theta) &= (-r, \theta \pm \pi) \text{ or } (r, \theta) = (-r, \theta \pm 180^\circ)\end{aligned}$$

EXAMPLE: Given the polar coordinate $(5, 300^\circ)$, find an equivalent polar coordinate that has the following characteristics: a.) $-360^\circ \leq \theta \leq 0$, $r > 0$ b.) $0 \leq \theta \leq 360^\circ$, $r < 0$, c.) $360^\circ \leq \theta \leq 720^\circ$, $r > 0$

a.) In this problem we are told to work in degrees. We want an angle that is negative that will lead us to the same point. We are allowed to add or subtract a 360° and that won't change our problem. So we can do $300^\circ - 360^\circ = -60^\circ$. This is an equivalent angle. So our equivalent point is $(5, -60^\circ)$. Our r is positive, so we are done.

b.) Now we want r to be negative. The formula above $(r, \theta) = (-r, \theta \pm 180^\circ)$ tells us that we can add a 180 degrees to our angle and this will change the r to a $-r$. Now if we add 180 degrees then we will get an angle more than 360 degrees, so we must subtract 180 degrees: $300^\circ - 180^\circ = 120^\circ$. Our equivalent point is: $(-5, 120^\circ)$.

c.) We want r to be positive and we need an angle that is more than one revolution, so we just need to add 360° to our angle: $300^\circ + 360^\circ = 660^\circ$. Our equivalent point is: $(5, 660^\circ)$.

EXAMPLE: Given the polar coordinate $\left(4, \frac{3\pi}{4}\right)$, find an equivalent polar coordinate that has the following characteristics: a.) $-2\pi \leq \theta \leq 0$, $r > 0$ b.) $0 \leq \theta \leq 2\pi$, $r < 0$, c.) $2\pi \leq \theta \leq 4\pi$, $r > 0$

a.) In this problem we are told to work in radians. We want an angle that is negative that will lead us to the same point. We are allowed to add or subtract a 2π and that won't change our problem. So we can do $\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4}$. This is an equivalent angle. So our equivalent point is $\left(4, -\frac{5\pi}{4}\right)$. Our r is positive, so we are done.

b.) Now we want r to be negative. The formula above $(r, \theta) = (-r, \theta \pm \pi)$ tells us that we can add a pi to our angle and this will change the r to a $-r$. Now if we subtract a pi then we will get a negative angle, and our question tells us we must have a positive angle, so we will add pi to our angle: $\frac{3\pi}{4} + \pi = \frac{7\pi}{4}$. Adding the pi will change our r to a negative r , so our equivalent point is: $\left(-4, \frac{7\pi}{4}\right)$.

c.) We want r to be positive and we need an angle that is more than one revolution, so we just need to add a 2π to our angle: $\frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$. Our equivalent point is: $\left(4, \frac{11\pi}{4}\right)$.

EXAMPLE: Given the polar coordinate $(-2, -120^\circ)$, find an equivalent polar coordinate that has the following characteristics: a.) $-360^\circ \leq \theta \leq 0$, $r > 0$ b.) $0 \leq \theta \leq 360^\circ$, $r < 0$, c.) $360^\circ \leq \theta \leq 720^\circ$, $r > 0$

a.) For this problem we are working all in degrees. We want an angle that is negative but an r that is positive. Our formula says that we can either add or subtract 180° from our original angle. If we add 180° then we won't have a negative angle anymore so we need to subtract: $-120^\circ - 180^\circ = -300^\circ$. Now we have the equivalent point: $(2, -300^\circ)$.

b.) We already have a negative r . Now we need a positive angle. We don't want to change r , so we need to add 360° to our original angle: $-120^\circ + 360^\circ = 240^\circ$. So our equivalent point is: $(-2, 240^\circ)$.

c.) We need r to be positive and our angle needs to be $360^\circ \leq \theta \leq 720^\circ$. We need to add 180° to change the r into a negative r . $-120^\circ + 180^\circ = 60^\circ$. We are not done yet because this is not between 360 degrees and 720 degrees. We can add 360 degrees and this won't change our r : $60^\circ + 360^\circ = 420^\circ$.

Conversion formulas from polar to rectangular coordinates.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

EXAMPLE: Convert $\left(5, \frac{\pi}{3}\right)$ into a rectangular point.

We can use the above formulas and plug in a 5 for r and a $\frac{\pi}{3}$ for θ . We will have: $x = 5 \cos \frac{\pi}{3}$. This equals:

$x = 5 \left(\frac{1}{2}\right) = \frac{5}{2}$. Now we will find y: $y = 5 \sin \frac{\pi}{3}$. This equals: $y = 5 \left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$. So our rectangular point is $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

EXAMPLE: Convert $\left(-3, -\frac{\pi}{4}\right)$ into a rectangular point.

We can use the above formulas and plug in a -3 for r and a $-\frac{\pi}{4}$ for θ . We will have: $x = -3 \cos\left(-\frac{\pi}{4}\right)$. This

equals: $x = -3 \left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$. Now we will find y: $y = -3 \sin\left(-\frac{\pi}{4}\right)$. This equals: $y = -3 \left(-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$.

So our rectangular point is $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

EXAMPLE: Convert $\left(-2, \frac{2\pi}{3}\right)$ into a rectangular point.

We can use the above formulas and plug in a -2 for r and a $\frac{2\pi}{3}$ for θ . We will have: $x = -2 \cos\left(\frac{2\pi}{3}\right)$. This

equals: $x = -2 \left(-\frac{1}{2}\right) = 1$. Now we will find y: $y = -2 \sin\left(\frac{2\pi}{3}\right)$. This equals: $y = -2 \left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$. So our

rectangular point is $(1, -\sqrt{3})$.

EXAMPLE: Convert the equation $r = 5 \sec \theta$ into a rectangular equation.

We can use a trig identity to write this as $r = \frac{5}{\cos \theta}$. We can cross multiply to get: $r \cos \theta = 5$. We can use the identity $x = r \cos \theta$ to get our final answer, $x = 5$, which is a vertical line.

EXAMPLE: Convert the equation $r = 4$ into a rectangular equation.

We can square both sides by r to get: $r^2 = 16$. So $x^2 + y^2 = 16$, which is a circle.

EXAMPLE: Convert the equation $r = 2\sin\theta - 4\cos\theta$ into a rectangular equation.

First let's multiply the both sides of the equation by r . This will allow us to put in substitutions for the sine and cosine: $r^2 = 2r\sin\theta - 4r\cos\theta$. Now we will replace the r^2 with $x^2 + y^2$ and we can replace the $r\sin\theta$ with y and the $r\cos\theta$ with x . Then we have: $x^2 + y^2 = 2y - 4x$. On the quiz and test this type of question will be given as multiple choice. You would notice that our answer above would not appear as one of the choices.

There is more we can do with this one. First set it equal to zero and group all the x and y terms together:

$$x^2 + 4x + y^2 - 2y = 0. \text{ Now we can complete the square on both sides: } x^2 + 4x + 4 + y^2 - 2y + 1 = 0 + 4 + 1.$$

Now we factor and our answer is: $(x+2)^2 + (y-1)^2 = 5$. This is a circle.

Conversion formulas from rectangular to polar coordinates

$$x^2 + y^2 = r^2$$

If (x, y) is in the first or fourth quadrant, then $\theta = \tan^{-1} \frac{y}{x}$.

If (x, y) is in the second or third quadrant, then $\theta = \tan^{-1} \frac{y}{x} + \pi$.

EXAMPLE: Convert $(-3, 3)$ into a polar coordinate. Express your angle in radians.

We can use the above formulas and plug in a -3 for x and a 3 for y . This will give us r : $(-3)^2 + (3)^2 = r^2$. This will give us $r = 3\sqrt{2}$. If we plot $(-3, 3)$ we will end up in the second quadrant. So we will use

$$\theta = \tan^{-1} \frac{y}{x} + \pi, \text{ so } \theta = \tan^{-1} \frac{3}{-3} + \pi. \text{ This equals: } \theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}. \text{ Polar coordinates are } \left(3\sqrt{2}, \frac{3\pi}{4} \right).$$

EXAMPLE: Convert $(-2, -2\sqrt{3})$ into a polar coordinate. Express your angle in radians.

We can use the above formulas and plug in a -2 for x and a $-2\sqrt{3}$ for y . This will give us r :

$(-2)^2 + (-2\sqrt{3})^2 = r^2$. This will give us $r^2 = 4 + 12$, so $r = 4$. If we plot $(-2, -2\sqrt{3})$ we will end up in the third quadrant. So we will use $\theta = \tan^{-1} \frac{y}{x} + \pi$, so $\theta = \tan^{-1} \frac{-2\sqrt{3}}{-2} + \pi$. This equals: $\theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$. So

our polar coordinates are $\left(4, \frac{4\pi}{3} \right)$.

EXAMPLE: Convert $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ into a polar coordinate. Express your angle in radians.

We can use the above formulas and plug in a $\frac{1}{2}$ for x and a $-\frac{\sqrt{3}}{2}$ for y. This will give us r:

$\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = r^2$. This will give us $r^2 = \frac{1}{4} + \frac{3}{4}$, so $r = 1$. If we plot $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ we will end up in the

fourth quadrant. So we will use $\theta = \tan^{-1} \frac{y}{x}$, so $\theta = \tan^{-1} \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$. This equals: $\theta = \tan^{-1} -\sqrt{3}$. Therefore,

$\theta = -\frac{\pi}{3}$. So our polar coordinates are $\left(1, -\frac{\pi}{3}\right)$.

EXAMPLE: Convert the equation $x^2 + y^2 = x$ into a polar equation and solve for r.

In this equation we will replace $x^2 + y^2$ with r^2 and we will replace x with $r \cos \theta$. We get: $r^2 = r \cos \theta$.

We need to set this equal to zero and solve for r. We will get: $r^2 - r \cos \theta = 0$. Now factor out an r: $r(r - \cos \theta) = 0$. Solving for r we get: $r = 0$ and $r = \cos \theta$.

EXAMPLE: Convert the equation $4x^2y = 1$ into a polar equation and solve for r.

In this equation we will replace x^2 with $r^2 \cos^2 \theta$ and we will replace y with $r \sin \theta$. We get:

$4r^2 \cos^2 \theta \cdot r \sin \theta = 1$. This equals $4r^3 \cos^2 \theta \cdot \sin \theta = 1$. Now we want to solve for r^3 by dividing:

$r^3 = \frac{1}{4 \cos^2 \theta \cdot \sin \theta}$. Now take the cube root of both sides: $r = \sqrt[3]{\frac{1}{4 \cos^2 \theta \cdot \sin \theta}}$. We can use identities here but it is not necessary.

EXAMPLE: Convert the equation $y^2 = 2x$ into a polar equation and solve for r.

In this equation we will replace y^2 with $r^2 \sin^2 \theta$ and we will replace x with $r \cos \theta$. You will get:

$r^2 \sin^2 \theta = 2r \cos \theta$. On the quiz and test this type of question will be given as multiple choice. You would notice that our answer above would not appear as one of our choices. That means we need to simplify this further. First set this equal to zero: $r^2 \sin^2 \theta - 2r \cos \theta = 0$. Now factor out an r: $r(r \sin^2 \theta - 2 \cos \theta) = 0$.

Now set both factors equal to zero. You will get $r = 0$ and $r \sin^2 \theta - 2 \cos \theta = 0$. We need to solve the second

equation for r. You will get: $r = \frac{2 \cos \theta}{\sin^2 \theta}$. This can be written as: $r = 2 \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta}$. Then our final answer

is $r = 2 \cot \theta \csc \theta$.