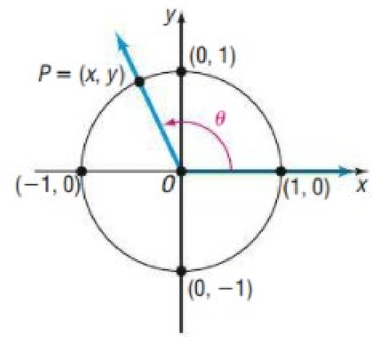
6.3 Properties of the Trigonometric Functions

In this section we will be by looking at the domain and range of the six trigonometric functions. We can get the domains from the unit circle. The ranges come from the graphs of these functions. We see that the x and y values are between -1 and 1. We can apply this to the individual trig functions. In the table below, *n* represents any integer.

Function	**Symbol**	**Domain**	**Range**
sine	$f(\theta)=\sin\theta$	$(-\infty,\infty)$	$[-1,1]$
cosine	$f(\theta)=\cos\theta$	$(-\infty,\infty)$	$[-1,1]$
tangent	$f(\theta)=\tan\theta$	$\left(-\infty,\frac{n\pi}{2}\right)\cup\left(\frac{n\pi}{2},\infty\right)$	$(-\infty,\infty)$
cosecant	$f(\theta)=\csc\theta$	$(-\infty,n\pi)\cup(n\pi,\infty)$	$(-\infty,-1]\cup[1,\infty)$
secant	$f(\theta)=\sec\theta$	$\left(-\infty,\frac{n\pi}{2}\right)\cup\left(\frac{n\pi}{2},\infty\right)$	$(-\infty,-1]\cup[1,\infty)$
cotangent	$f(\theta)=\cot\theta$	$(-\infty,n\pi)\cup(n\pi,\infty)$	$(-\infty,\infty)$



Even – Odd Properties

$\cos(-t)=\cos t$ $\qquad$ $\sec(-t)=\sec t$

$\sin(-t)=-\sin t$ $\qquad$ $\csc(-t)=-\csc t$

$\tan(-t)=-\tan t$ $\qquad$ $\cot(-t)=-\cot t$

EXAMPLE: Use the even-odd properties to find the exact value of $\cos(-30^\circ)$ without using a calculator.

Using the even-odd property we get $\cos(-30^\circ)=\cos(30^\circ)$. Then we can use our unit circle to find the exact value. Therefore, $\cos(-30^\circ)=\cos(30^\circ)=\frac{\sqrt{3}}{2}$.

EXAMPLE: Use the even-odd properties to find the exact value of $\csc\left(-\frac{\pi}{4}\right)$ without using a calculator.

Using the even-odd property we get $\csc\left(-\frac{\pi}{4}\right)=-\csc\left(\frac{\pi}{4}\right)$. Then we can use our unit circle to find the exact value. Therefore, $\csc\left(-\frac{\pi}{4}\right)=-\csc\left(\frac{\pi}{4}\right)=-\frac{1}{\sin\left(\frac{\pi}{4}\right)}=-\frac{1}{\frac{\sqrt{2}}{2}}=-\frac{2}{\sqrt{2}}=-\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=-\frac{2\sqrt{2}}{2}=-\sqrt{2}$.

Periodic Properties

If we start at an angle and go around one revolution (360° or 2π radians) we will end up at the same angle we started with. The *k* value is any integer, and represents how many revolutions are going around. If you want to use degrees, replace the $2\pi k$ in the equations below with 360*k*. For tangent and cotangent you will end up at the same spot if you add 2π, however you will also get the same value if you just add π.

$\sin(t \pm 2\pi k) = \sin t$ $\qquad$ $\csc(t \pm 2\pi k) = \csc t$

$\cos(t \pm 2\pi k) = \cos t$ $\qquad$ $\sec(t \pm 2\pi k) = \sec t$

$\tan(t \pm \pi k) = \tan t$ $\qquad$ $\cot(t \pm \pi k) = \cot t$

EXAMPLE: Find the EXACT value of $\sin\left(1485^\circ\right)$.

We need to divide 1485 by 360 to see how many revolutions we have. If you divide 1485 by 360 you will get 4 with a remainder of 45. So we can rewrite our problem as: $\sin\left(45^\circ + 360(4)\right)$. From our definitions above we know that $\sin\left(45^\circ + 360(4)\right) = \sin 45^\circ$. From our previous values we know that $\sin 45^\circ = \frac{\sqrt{2}}{2}$. So we know that $\sin 1485^\circ = \frac{\sqrt{2}}{2}$.

EXAMPLE: Find the EXACT value of $\tan\left(-\frac{17\pi}{4}\right)$.

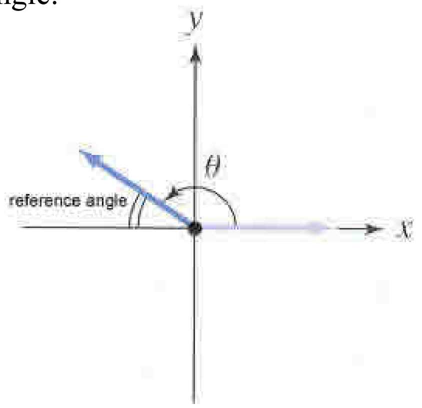
First we will use our even-odd property to change the negative angle into a positive angle: $\tan\left(-\frac{17\pi}{4}\right) = -\tan\left(\frac{17\pi}{4}\right)$. It would be easier to change this into degrees: $\frac{17\pi}{4} \cdot \frac{180}{\pi} = 765^\circ$. Dividing this by 360 we will get 2 with a remainder of 45. So our problem becomes $\tan\left(45^\circ + 360(2)\right)$. From our periodic properties, $\tan\left(45^\circ + 360(2)\right) = \tan 45^\circ$. From our previous values we know that $\tan 45^\circ = 1$. So we know that $-\tan\left(\frac{17\pi}{4}\right) = -1$.

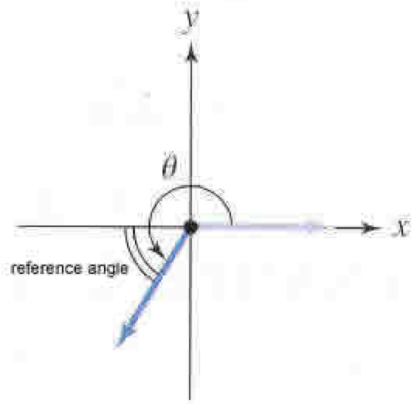
EXAMPLE: Use the Even-Odd Properties and Periodic Properties to simplify:
$-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t)$

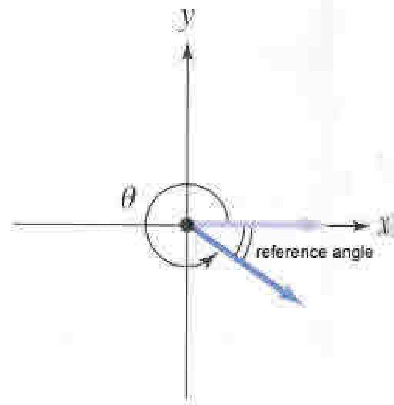
For the first term, since we are adding 2π we can say $-2\sin(3t + 2\pi) = -2\sin(3t)$ by applying the Periodic Properties. Next, we can say $-3\sin(-3t) = -3\left(-\sin(3t)\right) = 3\sin(3t)$ and $\cos(-2t) = \cos(2t)$ by the Even-Odd Properties.

So $-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t) = -2\sin(3t) + 3\sin(3t) + \cos(2t)$. The two sine terms are like terms, so $-2\sin(3t + 2\pi) - 3\sin(-3t) + \cos(-2t) = \sin(3t) + \cos(2t)$.

Reference Angle – an angle between 0 and 90 that is formed by the terminal side of an angle and the x-axis. The reference angle is labeled below. It is indicated by the double curved lines. Notice that no matter where the angle is drawn it is measured from the x-axis. Under each drawing it tells you how to find the reference angle:







If $90^\circ < \theta < 180^\circ$ then
Ref. angle = $180^\circ - \theta$

If $\frac{\pi}{2} < \theta < \pi$ then
Ref. angle = $\pi - \theta$

If $180^\circ < \theta < 270^\circ$ then
Ref. angle = $\theta - 180^\circ$

If $\pi < \theta < \frac{3\pi}{2}$ then
Ref. angle = $\theta - \pi$

If $270^\circ < \theta < 360^\circ$ then
Ref. angle = $360^\circ - \theta$

If $\frac{3\pi}{2} < \theta < 2\pi$ then
Ref. angle = $2\pi - \theta$

EXAMPLE: Find the reference angle for 170°.

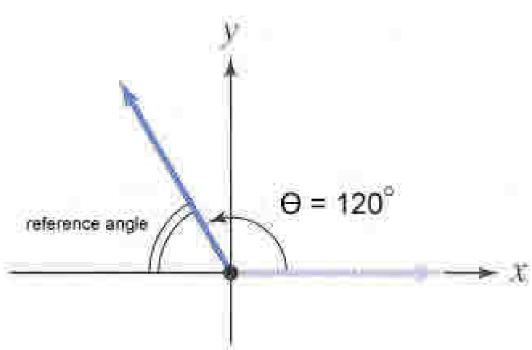
Since $90^\circ < \theta < 180^\circ$, we will use the formula $180^\circ - \theta$, so the reference angle is $180^\circ - 170^\circ = 10^\circ$.

EXAMPLE: Find the reference angle for $\frac{9\pi}{5}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, we will use the formula $2\pi - \theta$, so the reference angle is $2\pi - \frac{9\pi}{5} = \frac{10\pi}{5} - \frac{9\pi}{5} = \frac{\pi}{5}$.

EXAMPLE: Draw 120° in standard position and then find its reference angle.

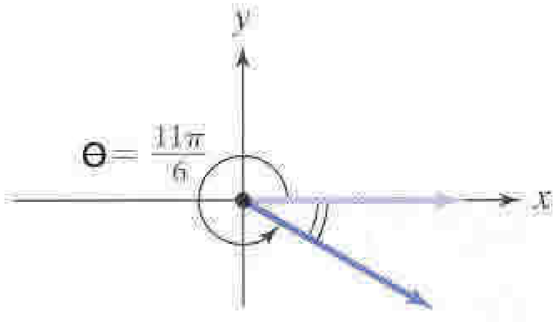
First we will draw it in standard position. The reference angle is indicated by the double curved lines:



To find the reference angle, we use the formula above, which says that the reference angle is $180^\circ - \theta$. So we our reference angle is: $180^\circ - 120^\circ = 60^\circ$.

EXAMPLE: Draw $\frac{11\pi}{6}$ in standard position and then find its reference angle.

We can change $\frac{11\pi}{6}$ into degrees so we know how to graph it: $\frac{11\pi}{6}\cdot\frac{180}{\pi}=330^\circ$. Now we will draw it in standard position. The reference angle is indicated by the double curved lines



To find the reference angle, we use the formula above, which says that the reference angle is $360^\circ-\theta$. So we our reference angle is: $360^\circ-330^\circ=30^\circ$. We need to change this back into radians since the problem was originally given in radians. Our reference angle is: $\frac{\pi}{6}$

Sign values of sine, cosine, and tangent in each quadrant

$\sin\theta$ + $\cos\theta$ − $\tan\theta$ −	$\sin\theta$ + $\cos\theta$ + $\tan\theta$ +
$\sin\theta$ − $\cos\theta$ − $\tan\theta$ +	$\sin\theta$ − $\cos\theta$ + $\tan\theta$ −

Depending on which quadrant you are in the sine, cosine, and tangent functions will be either positive or negative.

The quadrants are number from 1 to 4 counterclockwise starting with the upper right quadrant. Each quadrant has a certain angle value: In quadrant 1: $0<\theta<90^\circ$, in quadrant 2: $90<\theta<180$, in quadrant 3: $180<\theta<270$, and in quadrant 4: $270<\theta<360$.

An easy way to remember the sign chart is the phrase 'All Students Take Calculus' The first letter of each word in the phrase tells you what is positive in each quadrant, starting in quad. 1 and going counterclockwise.

ALL Means all of them are positive in the first quadrant
S Means sine is the only one positive in quad 2.
T Means tangent is the only one positive in quad 3
C Means cosine is the only one positive in quad 4

EXAMPLE: Name the quadrant in which the angle θ lies given $\sin\theta<0$ and $\cos\theta>0$.

This angle would be in quadrant IV based on the sign chart.

EXAMPLE: Name the quadrant in which the angle θ lies given $\tan\theta<0$ and $\cos\theta<0$.

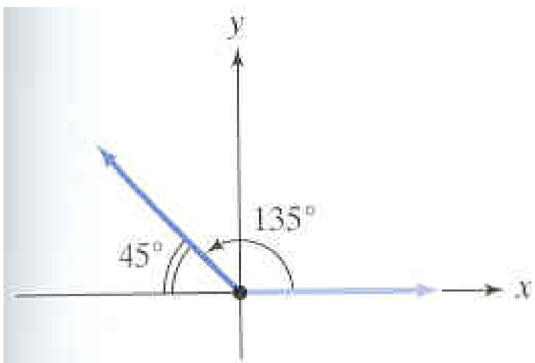
This angle would be in quadrant II based on the sign chart.

How to find the trigonometric value for any angle:

1.) Find the reference angle.
2.) Apply the trig function to the reference angle
3.) Apply the appropriate sign.

EXAMPLE: Find the exact value of $\cos 135^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We will follow the three steps from above.



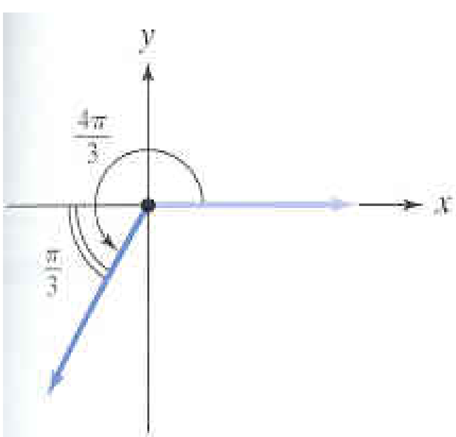
1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $180^\circ - 135^\circ = 45^\circ$

2.) We need to apply the trig function to our reference angle, so we will do $\cos 45^\circ = \frac{\sqrt{2}}{2}$.

3.) We need to apply the appropriate sign. This is where we will use the sign chart from the last page. This angle is in the second quadrant, so cosine needs to be negative here. So now we can write our answer: $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

EXAMPLE: Find the exact value of $\sin\frac{4\pi}{3}$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We can change this into degrees to see what quadrant we are in: $\frac{4\pi}{3}\cdot\frac{180}{\pi} = 240^\circ$

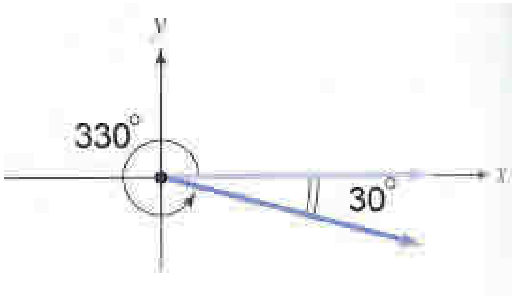


1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $240^\circ - 180^\circ = 60^\circ$. This is equivalent to $\frac{\pi}{3}$.

2.) We need to apply the trig function to our reference angle, so we will do $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the third quadrant, so sine needs to be negative here. So now we can write our answer: $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value of $\cos 330^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.



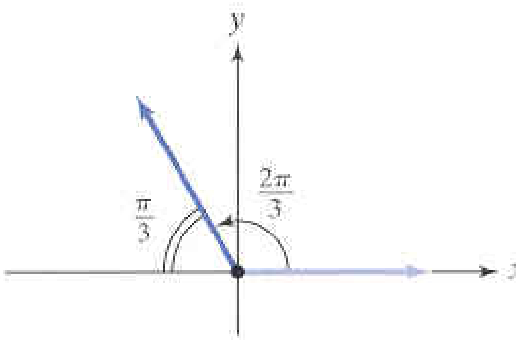
1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $360^\circ - 330^\circ = 30^\circ$.

2.) We need to apply the trig function to our reference angle, so we will do $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the fourth quadrant, so cosine needs to be positive here. So now we can write our answer: $\cos 330^\circ = \frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value of $\tan \frac{14\pi}{3}$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We can change this into degrees: $\frac{14\pi}{3} \cdot \frac{180}{\pi} = 840^\circ$. We can subtract two revolutions from this:
$840^\circ - 360^\circ - 360^\circ = 120^\circ$. From our rules, we can rewrite this problem as: $\tan(120^\circ + 360(2)) = \tan 120^\circ$
In radian form it would look like this: $\tan \frac{2\pi}{3}$.



1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. (180 – 120)

2.) We need to apply the trig function to our reference angle. We have $\tan \frac{\pi}{3} = \sqrt{3}$.

3.) We need to apply the appropriate sign. This angle is in the second quadrant, so tangent needs to be negative here. So now we can write our answer: $\tan \frac{14\pi}{3} = -\sqrt{3}$.

EXAMPLE: Find the reference angle of $\tan(-225^\circ)$ and find its EXACT value.

Using the even and odd properties, $\tan(-225^\circ) = -\tan 225^\circ$. We have changed the problem, so now we will look at positive 225 degrees. This is in the third quadrant, so our reference angle is $225^\circ - 180^\circ = 45^\circ$. We will now calculate $-\tan 45^\circ$. The value for $\tan 45^\circ$ is 1. So now we have $-\tan(45^\circ) = -1$ In the third quadrant, tangent is positive, so we don't need to change the sign of our answer, so $\tan(-225^\circ) = -1$.

EXAMPLE: Find the reference angle of $\sec(-210^\circ)$ and find its EXACT value.

Using the even and odd properties, $\sec(-210^\circ) = \sec 210^\circ$. We have changed the problem, so now we will look at positive 210 degrees. This is in the third quadrant, so our reference angle is $210^\circ - 180^\circ = 30^\circ$. We will do $\sec 30^\circ$. This is the same as $\frac{1}{\cos 30^\circ}$. Now we can put in the value off our table, and we will get $\frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Secant is negative in the 3rd quadrant, so we need to change the sign: $\sec(-210^\circ) = -\frac{2\sqrt{3}}{3}$.

EXAMPLE: Find the reference angle of $\sin\left(-\frac{11\pi}{3}\right)$ and find its EXACT value.

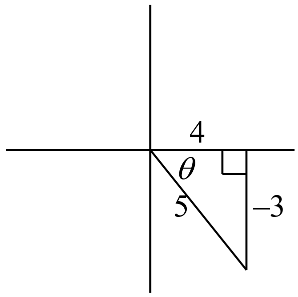
We want to change our angle into degrees: $-\frac{11\pi}{3} \cdot \frac{180}{\pi} = -660^\circ$. So now our problem becomes: $\sin(-660^\circ)$. Using the even and odd properties, $\sin(-660^\circ) = -\sin 660^\circ$. We have changed the problem, so now we will look at positive 660 degrees. We can rewrite the problem as $-\sin(300^\circ + 360^\circ)$ which becomes $-\sin(300^\circ)$. This is in the fourth quadrant, so our reference angle is $360^\circ - 300^\circ = 60^\circ$, or $\frac{\pi}{3}$. We will now look at $-\sin 60^\circ$. We can put in the value off our table, and we will get $-\frac{\sqrt{3}}{2}$. In the fourth quadrant, secant is negative, so we need to change the sign of our answer, so $\sin\left(-\frac{11\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

EXAMPLE: Find the reference angle of $\csc\left(-\frac{19\pi}{4}\right)$ and find its EXACT value.

Changing this into degrees we get: $-\frac{19\pi}{4} \cdot \frac{180}{\pi} = -855^\circ$. Using the even and odd properties, $\csc(-855^\circ) = -\csc 855^\circ$. This can be rewritten as $-\csc(135^\circ + 360^\circ(2))$. This reduces to $-\csc(135^\circ)$. This is in the second quadrant, so the reference angle is $180^\circ - 135^\circ = 45^\circ$. So now we have $-\csc 45^\circ$. This is the same as $-\dfrac{1}{\sin 45^\circ}$. From our table we have: $-\dfrac{1}{\frac{\sqrt{2}}{2}} = -\dfrac{2}{\sqrt{2}} = -\dfrac{2\sqrt{2}}{2} = -\sqrt{2}$. Since we are in the second quadrant, sine is positive, so we don't need to change our sign. Our answer is $\csc\left(-\dfrac{19\pi}{4}\right) = -\sqrt{2}$.

EXAMPLE: Given $\tan\theta = -\dfrac{3}{4}$ and $\sin\theta < 0$, find the exact value of the six trig functions.

This will be drawn in the fourth quadrant, so the opposite side must be negative. By the Pythagorean Theorem we find: $(-3)^2 + (4)^2 = c^2$. So c = 5. Remember the hypotenuse is ALWAYS positive.

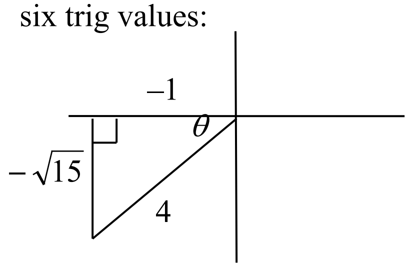


$\sin\theta = -\dfrac{3}{5}$, $\csc\theta = -\dfrac{5}{3}$, $\cos\theta = \dfrac{4}{5}$, $\sec\theta = \dfrac{5}{4}$, $\tan\theta = -\dfrac{3}{4}$, $\cot\theta = -\dfrac{4}{3}$

EXAMPLE: Given $\cos\theta = -\dfrac{1}{4}$ and $180^\circ < \theta < 270^\circ$, find the exact value of the six trig functions.

First we need to draw the triangle like we did in a previous section. This time we are told $180 < \theta < 270$, which means we need to draw the triangle in the third quadrant. Our fraction is negative. That means that either 1 or 4 must be negative when we put this in our drawing. The hypotenuse is NEVER negative, so this means that 1 must be negative since this is the adjacent side. Our θ is drawn at the origin, and this is always where it will be drawn. This is like a reference angle.

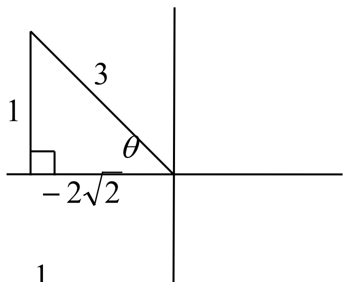
We can use the Pythagorean theorem to find the missing side: $a^2 + (-1)^2 = 4^2$. Solving this you will get $a = \pm\sqrt{15}$. In our drawing, since we are in the third quadrant, we MUST use the negative answer. The reason why is this vertical distance is really a y value, and if we think about it in terms of graphing something, the y would be negative since we are below the x-axis. So now our drawing is complete and we can find the six trig values:



$\sin\theta = -\dfrac{\sqrt{15}}{4}$, $\csc\theta = -\dfrac{4}{\sqrt{15}} = -\dfrac{4\sqrt{15}}{15}$, $\cos\theta = -\dfrac{1}{4}$, $\sec\theta = -4$, $\tan\theta = \sqrt{15}$, $\cot\theta = \dfrac{1}{\sqrt{15}} = \dfrac{\sqrt{15}}{15}$

EXAMPLE: Given $\csc\theta = 3$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of the six trig functions.

We are in the second quadrant. We can rewrite the original problem as $\csc\theta = \frac{3}{1}$. Now we know that the hypotenuse is 3 and the opposite side is 1. Using the Pythagorean theorem we can find the third side: $a^2 + 1^2 = 3^2$. Solving this we get $a = \pm\sqrt{8}$ which can be written as $a = \pm 2\sqrt{2}$. Since we are in the second quadrant we want to choose $-2\sqrt{2}$ since in the second quadrant the x value is negative.



$\sin\theta = \frac{1}{3}$, $\csc\theta = 3$, $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\sec\theta = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$, $\tan\theta = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$, $\cot\theta = -2\sqrt{2}$