6.4 Graphs of the Sine and Cosine Functions; Phase Shift

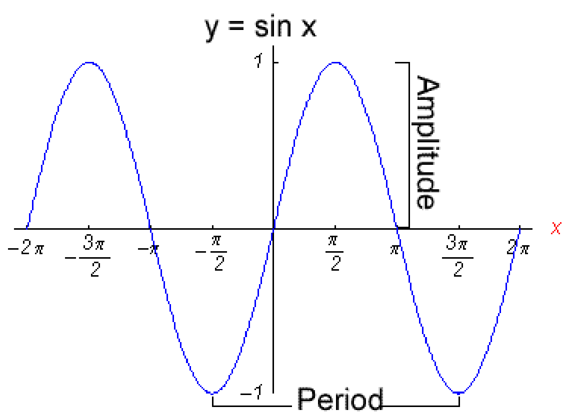
This section will introduce you to the sine and cosine functions. In order to generate each of these graphs, we can start with a table of values for the sine and cosine graph. Then we will plot the values below. We will look at specific angle since these provide easy to plot values. These specific angles are called **key points.** These values come directly from the unit circle.

$y = \sin\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin\theta$	0	1	0	–1	0

$y = \cos\theta$

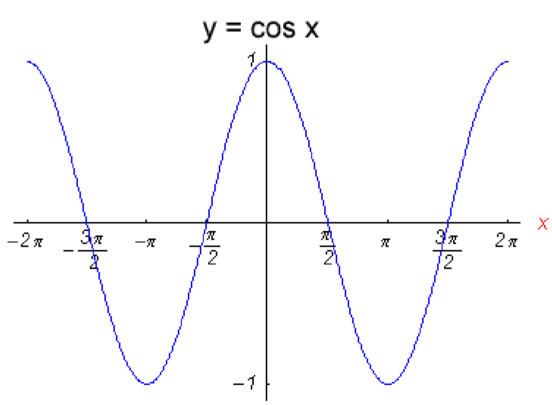
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos\theta$	1	0	–1	0	1



Period: How long it takes the graph to repeat itself
For sine graphs, the period is 2π .

Amplitude $= \dfrac{Highest\ \ value - Lowest\ \ value}{2}$

For the regular sine graph the amplitude is 1.



The period for cosine graphs is 2π

The amplitude for a regular cosine graph is 1.

General Form of a Sine or Cosine Equation:

$y = A\sin(Bx - C)$ or $y = A\cos(Bx - C)$

Amplitude = $|A|$, **Period** = $\frac{2\pi}{B}$, **Phase Shift** = $\frac{C}{B}$

The **phase shift** is a shift of the graph to the left or to the right. The direction depends on the sign of the phase shift:

If $\frac{C}{B} > 0$ the graph will shift to the right (this occurs when there is a minus sign between *Bx* and *C*).
If $\frac{C}{B} < 0$ the graph will shift to the left (this occurs when there is a plus sign between *Bx* and *C*).

The phase shift will always be one of the five key points. In the two regular graphs of sine and cosine, the phase shift is 0, That is why 0 is the starting key point of a cycle.

EXAMPLE: Indicate the amplitude, period, and phase shift without graphing: $y = -3.4\sin(5x - 7)$

First the amplitude is $|-3.4| = 3.4$. The period is $\frac{2\pi}{5}$. The phase shift is $\frac{7}{5}$.

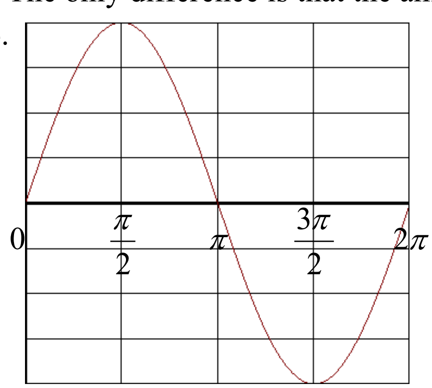
EXAMPLE: Indicate the period, amplitude, and phase shift without graphing: $y = -\frac{1}{5}\sin\left(\frac{\pi}{2}x + \frac{2\pi}{3}\right) + 1$.

The amplitude is $\left|-\frac{1}{5}\right|$ which is $\frac{1}{5}$. The period = $\frac{2\pi}{B}$, so this is $\frac{2\pi}{\frac{\pi}{2}}$. This simplifies to 4.

The phase shift is $\frac{-\frac{2\pi}{3}}{\frac{\pi}{2}} = -\frac{2\pi}{3} \cdot \frac{2}{\pi} = -\frac{4}{3}$.

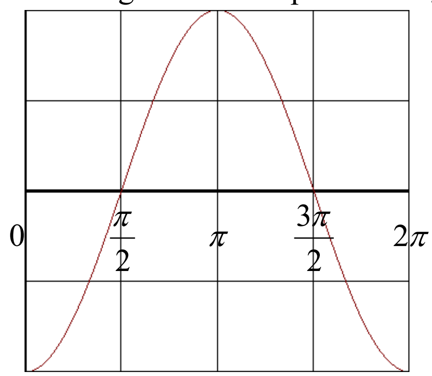
EXAMPLE: Graph over one period using transformations: $y = 4\sin x$

This will have the key points as $y = \sin x$. The only difference is that the amplitude is 4, so the highest and lowest points on the graph will be 4 and –4.



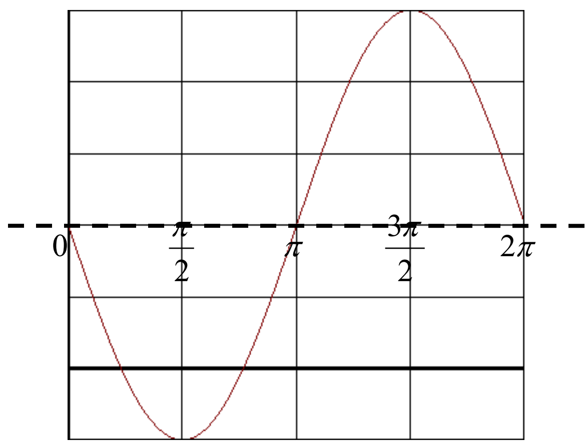
EXAMPLE: Graph over one period using transformations: $y = -2\cos x$

This one will have an amplitude of 2 and also the negative will flip over our graph. It will still have the same key points as $y = \cos x$.



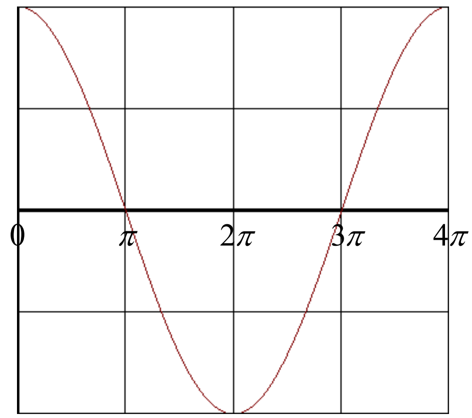
EXAMPLE: Graph over one period using transformations: $y = -3\sin x + 2$

This one will flip over the graph, raise the amplitude to 3, and then move the graph 2 units up. This will still have the same key points as $y = \sin x$. The only difference is that the axis is shifted up 2 units. The line y = 2 is called the **midline** since the line goes through the midline of the graph.

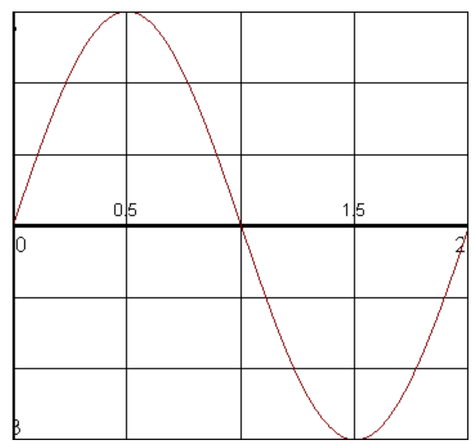


EXAMPLE: Graph over one period using transformations: $y = 2\cos\left(\frac{1}{2}x\right)$

Here the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. In this case the phase shift is 0, so the graph does not move vertically. In order to get the key points, we will use what is called the **quarter point**. The **quarter point** = $\frac{Period}{4}$. In our case the quarter point is $\frac{4\pi}{4} = \pi$. We start with the phase shift (0) and we keep adding π. The result is below. For cosine graphs, we always start at the number that is in front of the cosine. In this case it is 2, so at the phase shift the graph starts at 2.



EXAMPLE: Graph over one period using transformations: $y = 3\sin(\pi x)$



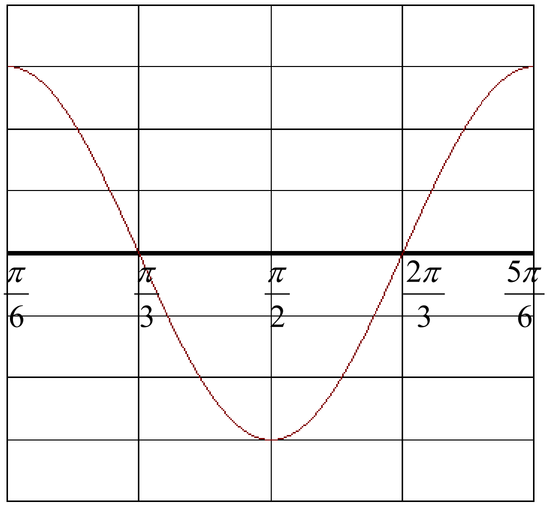
The period is $\frac{2\pi}{\pi} = 2$. The phase shift once again is 0. Now we want to find the quarter point, which is $\frac{2}{4} = \frac{1}{2}$. You many also use a decimal, which in this case is 0.5. We start at the phase shift (0) and keep adding 0.5 to get the other key points:
0 + 0.5 = 0.5, 0.5 + 0.5 = 1, 1 + 0.5 = 1.5, 1.5 + 0.5 = 2

EXAMPLE: Identify the amplitude, period, phase shift and graph of $y = 3\cos\left(3x - \frac{\pi}{2}\right)$. (Graph 1 period).

First the amplitude is $|3| = 3$. The period is $\frac{2\pi}{3}$. To find the phase shift, we take C and divide it by B. Then the phase shift is $\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$. This tells us the graph starts at $\frac{\pi}{6}$ because this is the phase shift. We need to find our 5 key points by finding the quarter point. In this problem, the quarter point is $\frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}$. We will start with the left key point $\frac{\pi}{6}$ and we will keep adding our quarter point to this to generate the other key points:

We start with $\frac{\pi}{6}$. Then we have: $\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}$, $\frac{2\pi}{6} + \frac{\pi}{6} = \frac{3\pi}{6}$, $\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}$, $\frac{4\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}$.

Now you can reduce each of your key points to the following: $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$ and then graph:



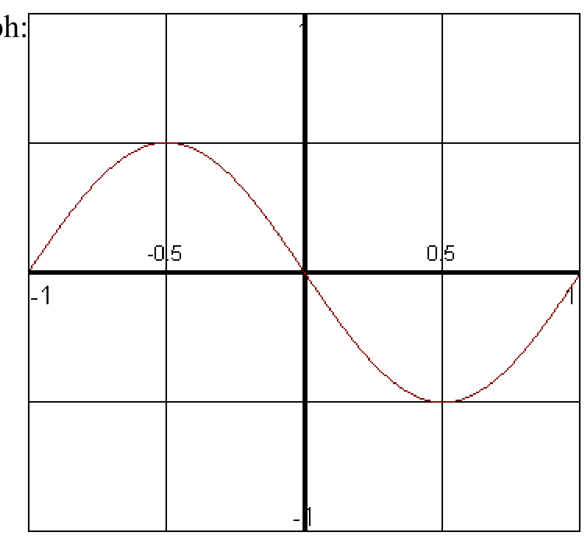
Remember the cosine graph always starts at the amplitude, which is 3 in this case.

EXAMPLE: Identify the amplitude, period, phase shift and graph of $y = \frac{1}{2}\sin(\pi x + \pi)$. (Graph 1 period).

First the amplitude is $\left|\frac{1}{2}\right| = \frac{1}{2}$. The period is $\frac{2\pi}{\pi} = 2$. Then the phase shift is $\frac{-\pi}{\pi} = -1$. This tells us the graph starts at -1. We need to find our 5 key points by finding the quarter point. The quarter point = $\frac{2}{4} = \frac{1}{2}$. We will start with the left key point 1 and we will keep adding our quarter point to this to generate the other key points:

We start with -1. Then we have: $-1 + \frac{1}{2} = -\frac{1}{2}$, $-\frac{1}{2} + \frac{1}{2} = 0$, $0 + \frac{1}{2} = \frac{1}{2}$, $\frac{1}{2} + \frac{1}{2} = 1$.

The key points are $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. We will put these on our graph:



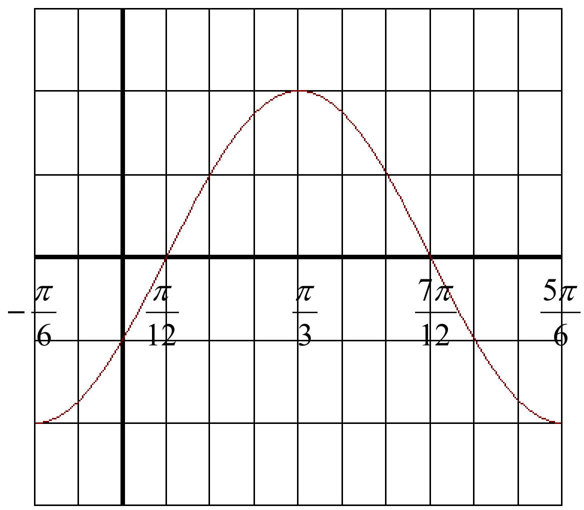
EXAMPLE: Identify the amplitude, period, phase shift and graph of $y = -2\cos\left(2x + \frac{\pi}{3}\right)$. (Graph 1 period).

First the amplitude is $|-2| = 2$. The period is $\frac{2\pi}{2} = \pi$. Then the phase shift is $\frac{\frac{-\pi}{3}}{2} = -\frac{\pi}{6}$. We need to find our 5 key points by finding the quarter point. The quarter point = $\frac{\pi}{4}$. We will start with the left key point 1 and we will keep adding our quarter point to this to generate the other key points:

We start with $-\frac{\pi}{6}$. Then we have: $-\frac{\pi}{6} + \frac{\pi}{4} = \frac{\pi}{12}$, $\frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$, $\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$, $\frac{7\pi}{12} + \frac{\pi}{4} = \frac{5\pi}{6}$.

The key points are $-\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}$. We will put these on our graph.

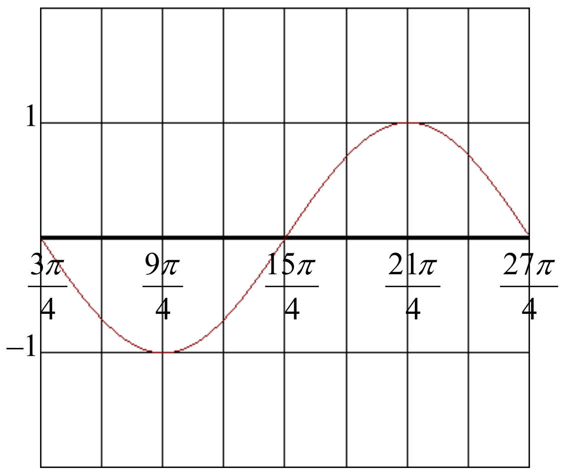
Notice we start this graph at -2 since there is a negative in our equation:



EXAMPLE: Identify the amplitude, period, phase shift and graph of $y = -\sin\left(\frac{1}{3}x - \frac{\pi}{4}\right)$. (Graph 1 period).

The amplitude is $|-1| = 1$. The period is $\frac{2\pi}{\frac{1}{3}} = 6\pi$. Then the phase shift is $\frac{\frac{\pi}{4}}{\frac{1}{3}} = \frac{3\pi}{4}$. We need to find our 5 key points by finding the quarter point. The quarter point = $\frac{6\pi}{4}$. There is no need to reduce this because we already notice that this has the same denominator as the phase shift. We can keep the same denominators to make it easier to add. We will start with the left key point $\frac{3\pi}{4}$ and we will keep adding our quarter point to this to generate the other key points: We start with $\frac{3\pi}{4}$. Then we have: $\frac{3\pi}{4}+\frac{6\pi}{4}=\frac{9\pi}{4}$, $\frac{9\pi}{4}+\frac{6\pi}{4}=\frac{15\pi}{4}$, $\frac{15\pi}{4}+\frac{6\pi}{4}=\frac{21\pi}{4}$, $\frac{21\pi}{4}+\frac{6\pi}{4}=\frac{27\pi}{4}$. The key points are $\frac{3\pi}{4}$, $\frac{9\pi}{4}$, $\frac{15\pi}{4}$, $\frac{21\pi}{4}$, $\frac{27\pi}{4}$.

We will put these on our graph:



EXAMPLE: Identify the amplitude, period, phase shift and graph of $-3\cos\left(-2x+\frac{\pi}{2}\right)$. (Graph 1 period).

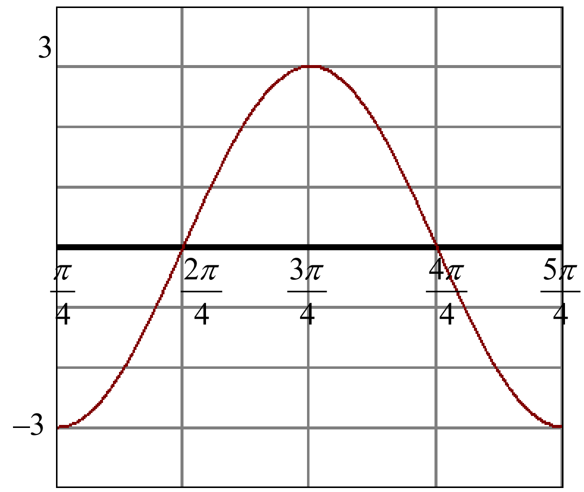
The formulas for the period and phase shift only work if the number in front of the x is positive. Before we can find the period and phase shift, we need to first factor out the negative: $-3\cos\left(-\left(2x-\frac{\pi}{2}\right)\right)$. Now we will use the even odd property: $\cos(-\theta)=\cos\theta$. Here, our angle $\theta = 2x-\frac{\pi}{2}$. This means the negative inside goes away, and the equation we will graph is $-3\cos\left(2x-\frac{\pi}{2}\right)$. NOTE: If this were a sine graph, you would use the even-odd property $\sin(-\theta)=-\sin\theta$.

The amplitude is $|-3|=3$. The period is $\frac{2\pi}{2}=\pi$. Then the phase shift is $\frac{\frac{\pi}{2}}{2}=\frac{\pi}{4}$. We need to find our 5 key points by finding the quarter point. The quarter point $=\frac{\pi}{4}$. We will start with the left key point 1 and we will keep adding our quarter point to this to generate the other key points:

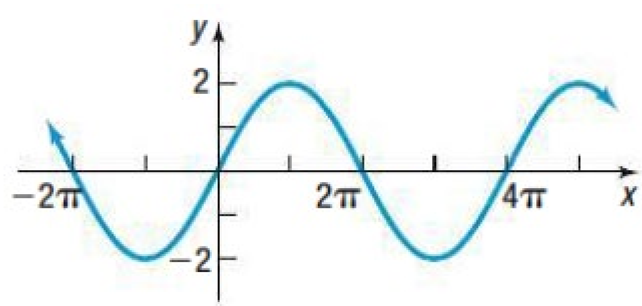
We start with $\frac{\pi}{4}$. Then we have: $\frac{\pi}{4}+\frac{\pi}{4}=\frac{2\pi}{4}$, $\frac{2\pi}{4}+\frac{\pi}{4}=\frac{3\pi}{4}$, $\frac{3\pi}{4}+\frac{\pi}{4}=\frac{4\pi}{4}$, $\frac{4\pi}{4}+\frac{\pi}{4}=\frac{5\pi}{4}$.

The key points are $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, $\frac{4\pi}{4}$, $\frac{5\pi}{4}$. We will put these on our graph.

Notice we start this graph at -3 since there is a negative in our equation:

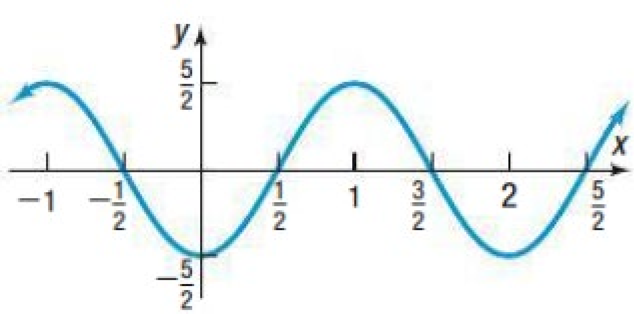


EXAMPLE: Find the equation of the given graph in the form $y = A\cos(Bx)$ or $y = A\sin(Bx)$.



We see that at $x = 0$ the graph is at 0. This means we need to use the equation $y = A\sin(Bx)$. At the second key point, the amplitude is 2, so $A = 2$. The graph crosses the x-axis at $x = 0$. Then it crosses the x-axis at $x = 4\pi$. This means the period is 4π. We will put this into period formula and solve for B: period $= \frac{2\pi}{B}$ so $4\pi = \frac{2\pi}{B}$ => $4\pi B = 2\pi$ => $B = \frac{1}{2}$. Therefore, our equation is: $y = 2\sin\left(\frac{1}{2}x\right)$

EXAMPLE: Find the equation of the given graph in the form $y = A\cos(Bx)$ or $y = A\sin(Bx)$.



We see that at $x = 0$ the graph is at the amplitude. This means we need to use the equation $y = A\cos(Bx)$. At the first key point, the graph is at -5/2, so A = -5/2. There is a valley is at $x = 0$. The next valley occurs at $x = 2$. This means the period is 2. We will put this into period formula and solve for B: period $= \frac{2\pi}{B}$, so $\frac{2}{1} = \frac{2\pi}{B}$ => $2B = 2\pi$ => $B = \pi$. Therefore, our equation is: $y = -\frac{5}{2}\cos(\pi x)$.