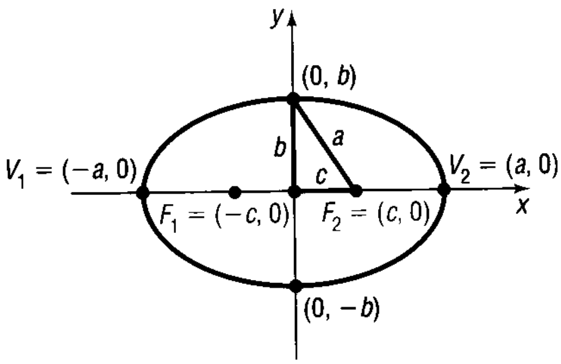
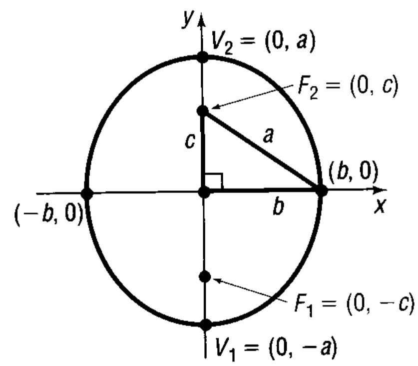
7.1 The Ellipse

In this section we will be looking at the ellipse which is basically an elongated circle. The pictures below show the two different ways an ellipse can be drawn that are both centered at the origin. The formulas below contain a and b. The a is the length of the major axis, or the longest axis. The b is the length of the shortest axis. It is very important to note that a is **ALWAYS** larger than b. If the larger number is under the x then the ellipse is drawn horizontally. If the larger number is under the y then the graph is drawn vertically.

Ellipses centered at (0, 0).





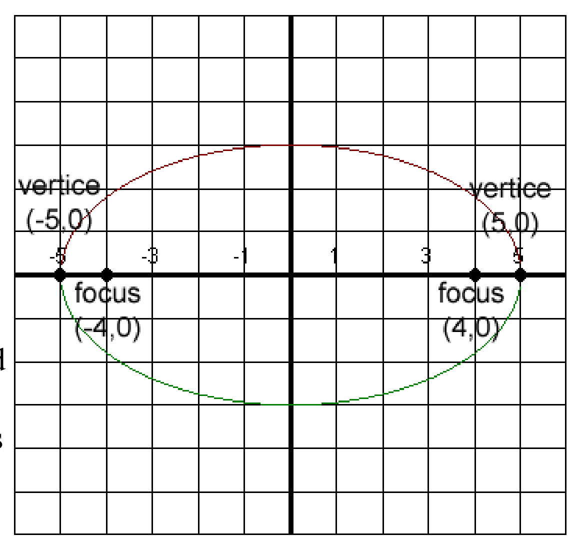
(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (b) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

In both of these cases the length of the major axis is 2a. The length of the minor axis is 2b. To find the c value in any of these graphs, use the equation $c = \sqrt{a^2 - b^2}$.

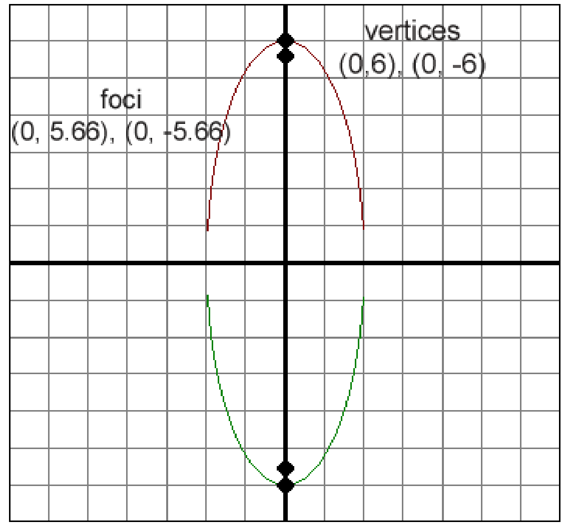
Eccentricity: a measure of how much the ellipse resembles a circle. The formula is $e = \frac{c}{a}$. If e = 0 then the ellipse is a circle. The larger the ellipse the skinnier it becomes.

EXAMPLE: Graph $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and identify the foci, eccentricity, center, length of the major axis, length of the minor axis, and the two vertices on the major axis.

The larger number is under the x, so we know this ellipse will be drawn horizontally. From our equation we know that a is 5 and b is 3. We can findc by the formula $c = \sqrt{a^2 - b^2}$: $c = \sqrt{5^2 - 3^2}$. So c = 4. We start at the center, which is (0, 0). Since this is drawn horizontally I will add 5 and subtract 5 from the x value of the center since a is 5. This will give us the vertices $(\pm 5,\ 0)$. Then I will go up and down 3 since this is our b. The foci run along the same line as the major axis, so from my center I will add 4 and subtract 4 from the x value of the center to find the foci. We find that the foci are at $(\pm 4,\ 0)$. We know the length of the major axis is 2(5) = 10. The length of the minor axis is 2(3) = 6. Finally we can find the eccentricity, which is $e = \frac{4}{5} = 0.8$.



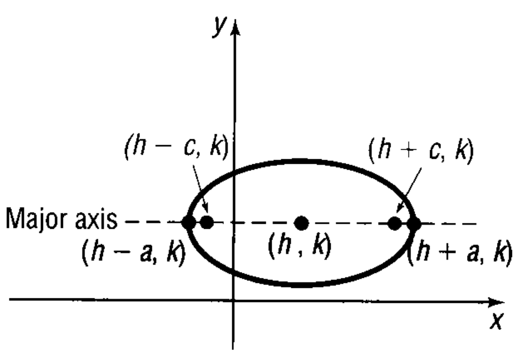
EXAMPLE: Graph $36x^2 + 4y^2 = 144$ and identify the foci, eccentricity, center, length of the major axis, length of the minor axis, and the two vertices on the major axis.



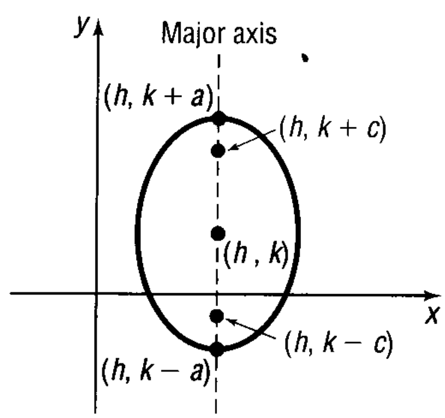
We need to first put this in the proper form. We need a 1 on the right hand side so we can divide the whole equation by 144 to get: $\frac{36x^2}{144} + \frac{4y^2}{144} = 1$. After reducing we get: $\frac{x^2}{4} + \frac{y^2}{36} = 1$.
The larger number is under the y, so we know this ellipse will be drawn vertically. From our equation we know that a is 6 and b is 2. We can find c by the formula $c = \sqrt{a^2 - b^2}$:
$c = \sqrt{6^2 - 2^2}$. So $c = \sqrt{32} = 4\sqrt{2} \approx 5.66$. We start at the center, which is (0, 0). Since this is drawn vertically I will add 6 and subtract 6 from the y value of the center since a is 6. This will give us the vertices $(0, \pm 6)$. Then I will go up and down 2 since this is our b. The foci run along the same line as the major axis, so from my center I will add 5.66 and subtract 5.66 from the y value of the center to find the foci. We find that the foci are at $(0, \pm 5.66)$.We know the length of the major axis is 2(6) = 12. The length of the minor axis is 2(2) = 4.
Finally we can find the eccentricity, which is $e = \frac{5.66}{6} = 0.94$.

Now we will look at ellipses that are centered at (h, k). Once again if the larger number is under the x then it will be drawn horizontally. If the larger number is under the y then it will go vertically. The pictures below include the formulas to find the foci and the vertices.

Ellipses centered at (h, k).



(a) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

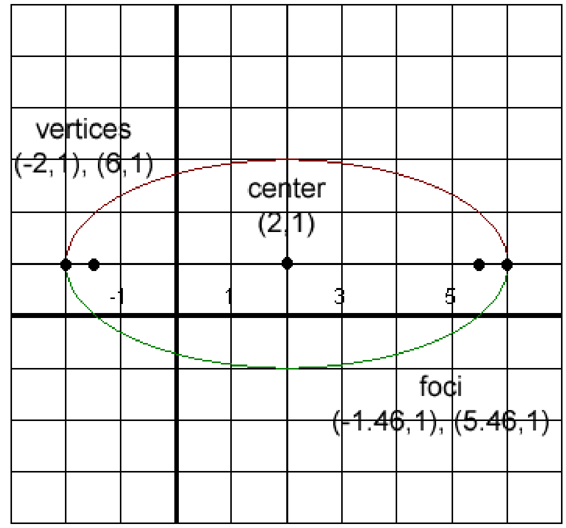


(b) $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

Once again in both of these cases the length of the major axis is 2a. The length of the minor axis is 2b. To find the c value in any of these graphs, use the equation $c = \sqrt{a^2 - b^2}$.

The eccentricity is still $e = \frac{c}{a}$.

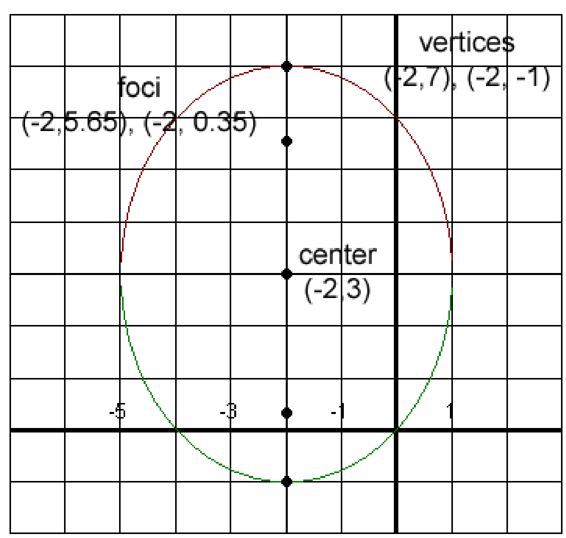
EXAMPLE: Graph $\frac{(x-2)^2}{16}+\frac{(y-1)^2}{4}=1$ and identify the foci, eccentricity, center, length of the major axis, length of the minor axis, and the two vertices on the major axis.



The larger number is under the x, so we know this ellipse will be drawn horizontally. From our equation we know that a is 4 and b is 2. We can find c by the formula $c=\sqrt{a^2-b^2}$: $c=\sqrt{4^2-2^2}$. So $c=\sqrt{12}=2\sqrt{3}\approx 3.46$. We start at the center, which is (2, 1). Remember to take the opposite sign of what is given. Since this is drawn horizontally I will add 4 and subtract 4 from the x value of the center since a is 4. This will give us the vertices (-3, 1) and (6, 1). Then I will go up and down 3 from the y value of our vertex since this is our b. The foci run along the same line as the major axis, so from my center I will add 3.46 and subtract 3.46 from the x value of the center to find the foci. We can write the foci as $(2\pm 2\sqrt{3},\ 0)$. We know the length of the major axis is 2(4) = 8. The length of the minor axis is 2(2) = 4. Finally we can find the eccentricity, which is $e=\frac{3.46}{4}=0.87$.

EXAMPLE: Graph $16x^2+9y^2+64x-54y+1=0$ and identify the foci, eccentricity, center, length of the major axis, length of the minor axis, and the two vertices on the major axis.

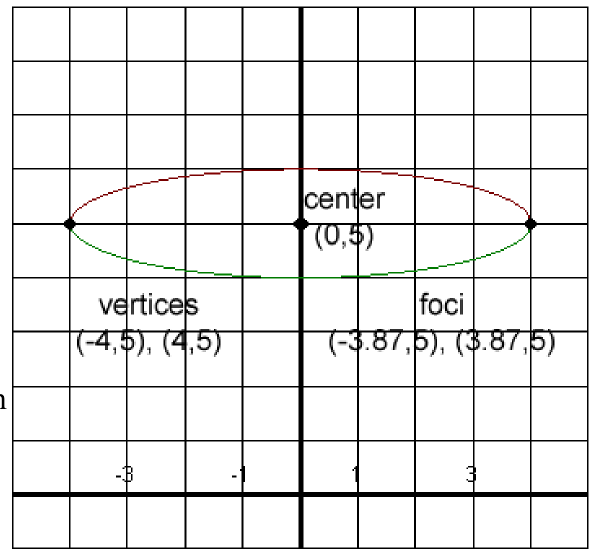
First we need to get this into the correct form. The first thing to do is subtract the 1 from both sides and group the x and y terms together: $16x^2+64x+9y^2-54y=-1$. Now I will factor the two x terms and the two y terms separately: $16(x^2+4x)+9(y^2-6y)=-1$. Now it is time to complete the square. In the first term we will divide the 4 by 2 and square it. You will get 4. We will add 4 to the left hand side however we do NOT add 4 to the right side. The 4 is really being multiplied by the 16, so we will add 4(16) = 64 to both sides. In the second term we will divide the 6 by 2 and square it. You will get 9. We will add 9 to the left hand side however we do NOT add 9 to the right side. The 9 is really being multiplied by the 9, so we will add 9(9) = 81 to both sides: $16(x^2+4x+4)+9(y^2-6y+9)=-1+64+81$. We can simplify the right side and factor the left side: $16(x+2)^2+9(y-3)^2=144$. Now divide both sides by 144 and reduce to get: $\frac{(x+2)^2}{9}+\frac{(y-3)^2}{16}=1$.



The larger number is under the y, so we know this ellipse will be drawn vertically. From our equation we know that a is 4 and b is 3. We can find c by the formula $c=\sqrt{a^2-b^2}$: $c=\sqrt{4^2-3^2}$. So $c=\sqrt{7}\approx 2.65$. We start at the center, which is (-2, 3). Since this is drawn vertically I will add 4 and subtract 4 from the y value of the center since a is 4. This will gives us the vertices (-2, 7) and (-2, -1). Then I will go left and right 3 since this is our b. The foci run along the same line as the major axis, so from my center I will add 2.65 and subtract 2.65 from the y value of the center to find the foci. We find that the foci are at $(-2,\ 3\pm\sqrt{7})$.We know the length of the major axis is 2(4) = 8. The length of the minor axis is 2(3) = 6. The eccentricity is $e=\frac{2.65}{4}\approx 0.66$.

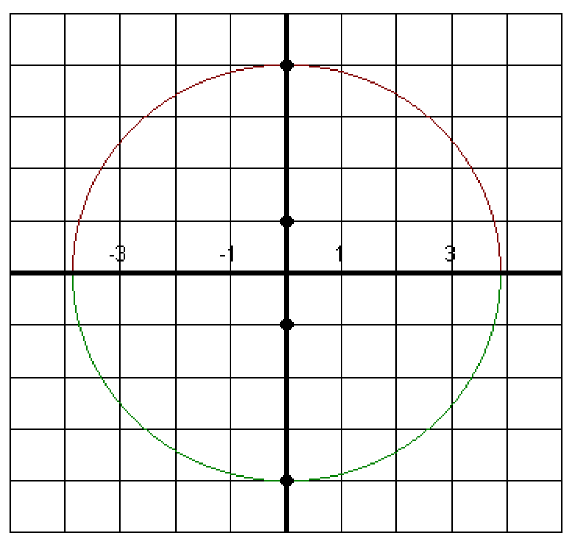
EXAMPLE: Graph $x^2 + 16y^2 - 160y + 384 = 0$ and identify the foci, eccentricity, center, length of the major axis, length of the minor axis, and the two vertices on the major axis.

First we need to get this into the correct form. The first thing to do is subtract the 384 from both sides and group the x and y terms together: $x^2 + 16y^2 - 160y = -384$. Now I will factor the two x terms and the two y terms separately: $x^2 + 16(y^2 - 10y) = -384$. Now it is time to complete the square. We can't do anything with the first term. In the second term we will divide the 10 by 2 and square it. You will get 25. We will add 25 to the left hand side however we do NOT add 25 to the right side. The 25 is really being multiplied by the 16, so we will add 16(25) = 400 to both sides: $x^2 + 16(y^2 - 10y + 25) = -384 + 400$. We can simplify the right side and factor the left side. I can rewrite the x as the quantity x – 0: $(x-0)^2 + 16(y-5)^2 = 16$. Now divide both sides by 16 and reduce to get: $\frac{(x-0)^2}{16} + \frac{(y-5)^2}{1} = 1$.



The larger number is under the x, so we know this ellipse will be drawn horizontally. From our equation we know that a is 4 and b is 1. We can find c by the formula $c = \sqrt{a^2 - b^2}$: $c = \sqrt{4^2 - 1^2}$. So $c = \sqrt{15} \approx 3.87$. We start at the center, which is (0, 5). Since this is drawn horizontally I will add 4 and subtract 4 from the x value of the center since a is 4. This will gives us the vertices (-4, 5) and (4, 5). Then I will add and subtract 1 from the y coordinate of our vertex since this is our b. The foci run along the same line as the major axis, so I will add 3.87 and subtract 3.87 from the x value of the center to find the foci. We find that the foci are at $(\pm\sqrt{15},\ 5)$.We know the length of the major axis is 2(4) = 8. The length of the minor axis is 2(1) = 2. The eccentricity is $e = \frac{3.87}{4} \approx 0.97$.

EXAMPLE: Find the equation of an ellipse with foci $(0,\ \pm 1)$ and vertices $(0,\ \pm 4)$.



First let's plot the above points. From this we can conclude that the ellipse must be centered at the origin. We also know that c = 1 because this is how far the foci are from the center. We also know a = 4 since that is how far the vertices are from the center. We can use $c = \sqrt{a^2 - b^2}$ to find b^2: $1 = \sqrt{4^2 - b^2}$. After squaring both sides we get: $1 = 16 - b^2$. Solving for b^2 we get $b^2 = 15$. Now we have everything we need to find our equation. From our graph we know this ellipse is drawn vertically, so it must use the formula $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Plugging in our known variables will give us: $\frac{x^2}{15} + \frac{y^2}{16} = 1$.

EXAMPLE: Find the equation of an ellipse centered at the origin with an eccentricity of $\frac{3}{5}$ and with one vertice at (-5, 0).

If the eccentricity is $\frac{3}{5}$ then we automatically know that c is 3 and a is 5. We can use $c=\sqrt{a^2-b^2}$ to find b: $3=\sqrt{5^2-b^2}$. Again we square both sides: $9=25-b^2$. Solving for b^2 we get $b^2=16$. If the vertice is (-5, 0) then this is 5 units to the left of the origin, which means the ellipse is drawn horizontally, so we can use $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Plugging in our unknowns will give us: $\frac{x^2}{25}+\frac{y^2}{16}=1$. We don't need a graph for this one.

EXAMPLE: Find the equation of an ellipse with foci at (1, 2) and (-3, 2) and with one vertex at (-4, 2).

First let's plot the above points. From this we can conclude that the ellipse must be centered at (-1, 2) since this point is halfway between our foci. We also know that c = 2 because this is how far the foci are from the center. We also know a = 3 since that is how far the vertices are from the center. We can use $c=\sqrt{a^2-b^2}$ to find b^2: $2=\sqrt{3^2-b^2}$. After squaring both sides we get: $4=9-b^2$. Solving for b^2 we get $b^2=5$. Now we have everything we need to find our equation. From our graph we know this ellipse is drawn horizontally, so it must use the formula $\frac{(x-h)^2}{b^2}+\frac{y-k)^2}{a^2}=1$. Plugging in our known variables will give us: $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{5}=1$.