7.1 The Inverse Sine, Cosine, and Tangent Functions

From our tables in a previous section we know that $\sin 30^\circ = \frac{1}{2}$. We put in an angle and get a value as a result.

In inverse trig functions we put in the value and get an angle: $\sin^{-1}\frac{1}{2} = 30^\circ$. So here we put in the value of one half and got 30 degrees as a result. We are not allowed to put any number into our inverse trig functions. There are restrictions on the domain that are given in the following table:

	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

NOTE: in some textbooks, the inverse functions are written differently, for example instead of $y = \sin^{-1} x$, some textbooks may write this as $y = \arcsin x$. So instead of the $^{-1}$ symbol, it is replaced by the word *arc*. These two mean exactly the same thing. So $y = \arccos x$ would mean the same as $y = \cos^{-1} x$, etc.

EXAMPLE: Find the $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{2}$." If you look on your table of values, go to the sine column and go down until you see the value $\frac{\sqrt{3}}{2}$. This corresponds to an angle of 60 degrees, which is the answer.

EXAMPLE: Find the $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

What this is really asking is: "find an angle between 0 and π that has a value of $\frac{\sqrt{2}}{2}$." If you look on your table of values, go to the cosine column and go down until you see the value $\frac{\sqrt{2}}{2}$. This also corresponds to an angle of 45 degrees, which is the answer.

EXAMPLE: Find the $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{3}$." If you look on your table of values, go to the tangent column and go down until you see the value $\frac{\sqrt{3}}{3}$. This also corresponds to an angle of 30 degrees, which is the answer.

EXAMPLE: Use a calculator to find $\cos^{-1} 0.7$, if possible, where $0 < \theta < 2\pi$. Round your answer to two decimal places.

We need to make sure our calculator is in radian mode before we proceed. The inverse cosine is above the cosine key on your calculator. You will probably need to use your second key in order to get the inverse cosine. Your answer should be 0.8. If you got an error, try entering 0.7 first and then get the inverse.

EXAMPLE: Use a calculator to find $\sin^{-1}(-1.2)$, if possible, where $0 < \theta < 2\pi$. Round your answer to two decimal places.

If you try putting this in your calculator you will get an error. This is because 1.2 is not in our domain. Recall that the domain for the inverse sine function is $-1 \le x \le 1$. This means we can only put in numbers between -1 and 1. So the answer is no solution.

Inverses and canceling

If we take $\cos^{-1}(\cos x)$ what will we get? Well, the inverse cosine and cosine will cancel and that will leave us with just x. However there are some restrictions on what x can be as listed below:

$$\cos^{-1}(\cos x) = x \quad \text{if} \quad 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \quad \text{if} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\sin(\sin^{-1} x) = x \quad \text{if} \quad -1 \le x \le 1$$
$$\tan^{-1}(\tan x) = x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\tan(\tan^{-1} x) = x \quad \text{if} \quad -\infty < x < \infty$$

EXAMPLE: Find the exact value if possible: $\tan(\tan^{-1} 5.3)$.

According to our restrictions above, x can be any number, so $\tan(\tan^{-1} 5.3) = 5.3$.

EXAMPLE: Find the exact value if possible: $\sin\left(\sin^{-1}\frac{98}{99}\right)$.

The fraction 98/99 is .9898, and this is less than 1, so $\sin\left(\sin^{-1}\frac{98}{99}\right) = \frac{98}{99}$.

EXAMPLE: Find the exact value if possible: $\cos\left(\cos^{-1}\sqrt{2}\right)$.

The square root changed into a decimal is 1.41, which is bigger than 1, so the answer is no solution since 1.41 is not in our domain.

EXAMPLE: Find the exact value: $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$

Since $\frac{\pi}{3}$ is in the domain $0 \le y \le \pi$, then our properties tell us $\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$.

EXAMPLE: Find the exact value: $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Be careful on this one. The answer is not $\frac{2\pi}{3}$. This is because $\frac{2\pi}{3}$ is not in $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. So we need to evaluate inside the parenthesis first, and then take the inverse. So we can use either a unit circle or reference angles to find $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. So now our problem becomes $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. We want to find an angle in the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ that has a y value of $\frac{\sqrt{3}}{2}$. Since sine is positive in the first quadrant, our answer is $\frac{\pi}{3}$. So $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

EXAMPLE: Find the exact value: $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$.

The answer is not $\frac{5\pi}{6}$. This is because $\frac{5\pi}{6}$ is not in $-\frac{\pi}{2} < y < \frac{\pi}{2}$. So we need to evaluate inside the parenthesis first, and then take the inverse. So we can use either a unit circle or reference angles to find $\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$. So now our problem becomes $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$. We want to find an angle in the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ that has a y value of $-\frac{\sqrt{3}}{3}$ (1st or 4th quadrant). Since this value is negative, we will look in the fourth quadrant. This value will occur at $\frac{11\pi}{6}$, however this angle is not on the interval $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. In order to make this in the correct interval you must subtract 2π: $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$. So $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = -\frac{\pi}{6}$.

EXAMPLE: Find the exact value: $2\cos^{-1} x = \pi$.

In order to solve this we need to isolate the inverse trig function, so we will first divide both sides by 2:

$\cos^{-1} x = \frac{\pi}{2}$. Now in order to cancel out the inverse cosine we need to take the cosine of both sides:

$\cos\left(\cos^{-1} x\right) = \cos\left(\frac{\pi}{2}\right)$. On the left side the cosine and inverse cosine will cancel.

$x = \cos\left(\frac{\pi}{2}\right) = 0$ To get the value of $\cos\left(\frac{\pi}{2}\right)$, we need to look on the table, so $x = 0$.

EXAMPLE: Find the exact value: $5\sin^{-1} x - 2\pi = 2\sin^{-1} x - 3\pi$.

We need to get all the inverse trig functions on one side of the equation, so we will first subtract the 2 inverse sine from both sides:

$3\sin^{-1} x - 2\pi = -3\pi$ Now add the 2 pi to both sides.

$3\sin^{-1} x = -\pi$ Divide both sides by 3.

$\sin^{-1} x = -\dfrac{\pi}{3}$ Now take the sine of both sides.

$\sin\left(\sin^{-1} x\right) = \sin\left(-\dfrac{\pi}{3}\right)$ The sine and inverse sine cancel from the left side

$x = \sin\left(-\dfrac{\pi}{3}\right)$ This is the same as negative 60 degrees. From our table, $\sin 60^\circ$ is $\dfrac{\sqrt{3}}{2}$.

$x = -\dfrac{\sqrt{3}}{2}$ The answer is negative because negative 60 degrees is in the 4th quadrant, so sine is negative.