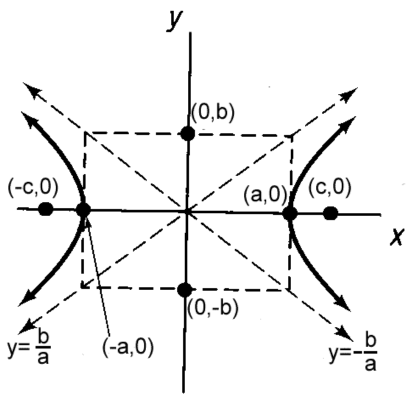
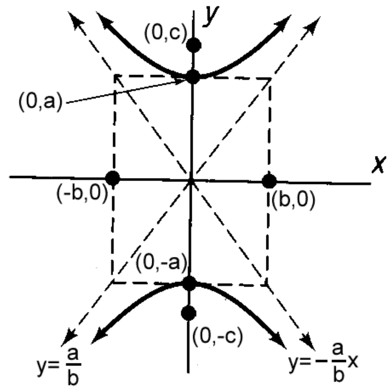
7.2 The Hyperbola

In this section we will be looking at the hyperbola whose shape is shown below. The pictures below show the two different ways a hyperbole can be drawn that are both centered at the origin. The equations contain a and b. You will use this to create a box as shown. The boxes are used as an aid in graphing the asymptotes. This time it does not matter if a or b is bigger. If the equation starts with x then it opens to the right and left then it is drawn horizontally. If the equation starts with y then it opens up and down and up then it is drawn vertically. **The first number in the denominator of the equation is always a regardless if x or y comes first.**

Hyperbolas centered at (0, 0).



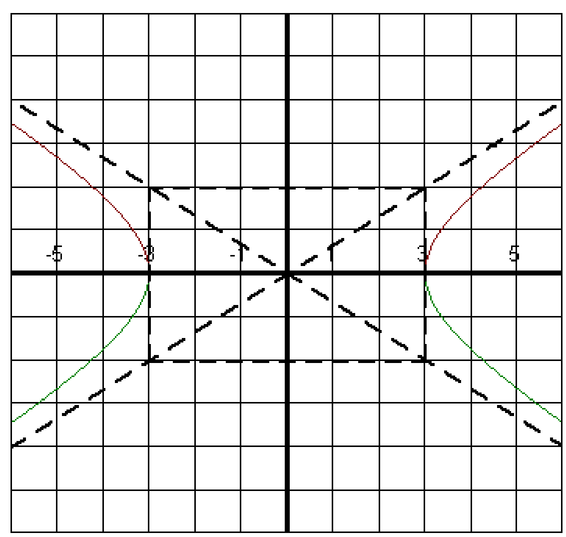


(a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Asymptotes: $y=\pm\frac{b}{a}x$

(b) $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ Asymptotes: $y=\pm\frac{a}{b}x$

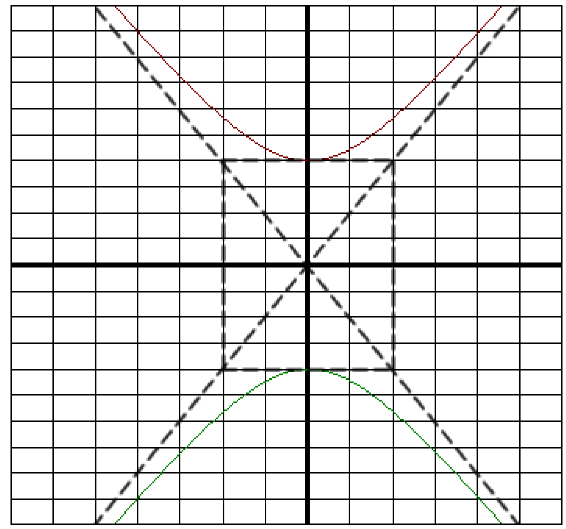
In both of these cases the length of the **transverse** axis is 2a. The length of the **conjugate** axis is 2b. To find the c value in any of these graphs, use the equation $c=\sqrt{a^2+b^2}$. This is used to find the foci. To find the **eccentricity** use the formula is $e=\frac{c}{a}$. The larger the e the wider the hyperbola is.

EXAMPLE: Graph $\frac{x^2}{9}-\frac{y^2}{4}=1$ and identify the foci, eccentricity, center, length of the transverse and conjugate axis, vertices, and the equations of the asymptotes.



Since x comes first in the equation, we know this hyperbola will be drawn horizontally. From our equation we know that a is 3 and b is 2. We can find c by the formula $c=\sqrt{a^2+b^2}$: $c=\sqrt{3^2+2^2}$. So $c=\sqrt{13}\approx 3.61$. We start at the center, at (0, 0). Since this is drawn horizontally I will add 3 and subtract 3 from the x value of the center since a is 3. This will give us the vertices $(\pm 3,\ 0)$. Then I will go up and down 2 since this is our b. This gives us the box. We can connect the corners of the box to create the asymptotes. The foci run along the same line as the transverse axis, so from my center I will add 3.61 and subtract 3.61 from the x value of the center to find the foci. The foci are at $(\pm\sqrt{13},\ 0)$. We know the length of the transverse axis is 2(3) = 6. The length of the conjugate axis is 2(2) = 4. Finally we can find the eccentricity, which is $e=\frac{3.61}{3}\approx 1.2$. The equation of the asymptotes are: $y=\pm\frac{2}{3}x$.

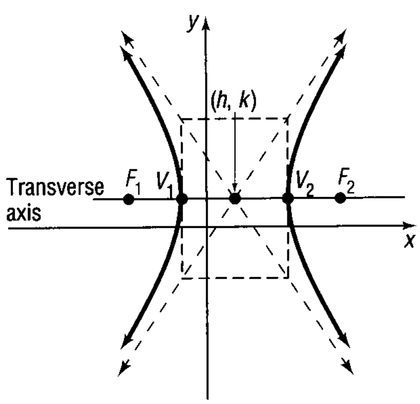
EXAMPLE: Graph $4y^2 - 16x^2 = 64$ and identify the foci, eccentricity, center, length of the transverse and conjugate axis, vertices, and the equations of the asymptotes.



We need to first put this in the proper form. We need a 1 on the right hand side so we can divide the whole equation by 64 to get: $\frac{4y^2}{64} - \frac{16x^2}{64} = 1$. After reducing we get: $\frac{y^2}{16} - \frac{x^2}{4} = 1$.
The y comes first, so we know this hyperbola will be drawn vertically. The a is always first, so a = 4 and b = 2. We can find c by the by the formula $c = \sqrt{a^2 + b^2}$: $c = \sqrt{4^2 + 2^2}$.
So $c = \sqrt{20} = 2\sqrt{5} \approx 4.47$. We start at the center, which is (0, 0) Since this is drawn vertically I will add 4 and subtract 4 from the y value of the center since a is 4. This will give us the vertices $(0, \pm 4)$ Then I will go to the left and right 2 since this is our b. This creates the box in which I can connect the corners to graph the asymptotes. The foci run along the same line as the major axis, so from my center I will add 4.47 and subtract 4.47 from the y value of the center to find the foci. We find that the foci are at $(0, \pm 2\sqrt{5})$ .We know the length of the transverse axis is 2(4) = 8. The length of the conjugate axis is 2(2) = 4.
Finally we can find the eccentricity, which is $e = \frac{4.47}{4} \approx 1.12$. The equation of the asymptotes are: $y = \pm \frac{4}{2} x$.
This reduces to $y = \pm 2x$.

Now we will look at hyperbolas that are centered at (h, k). Once again if the x comes first then it will be drawn horizontally. If y comes first it will go vertically. The pictures below include the formulas to find the foci and the vertices.

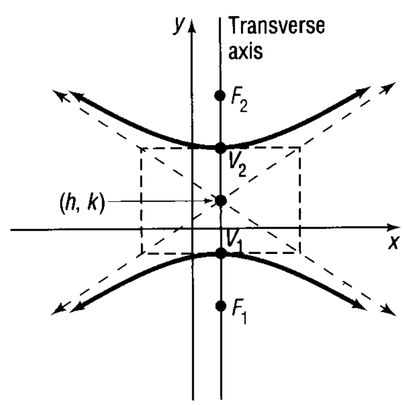
Hyperbolas centered at (h, k).



(a) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Asymptotes: $y - k = \pm \frac{b}{a}(x - h)$

Vertices: $(h \pm a,\ k)$, Foci: $(h \pm c,\ k)$



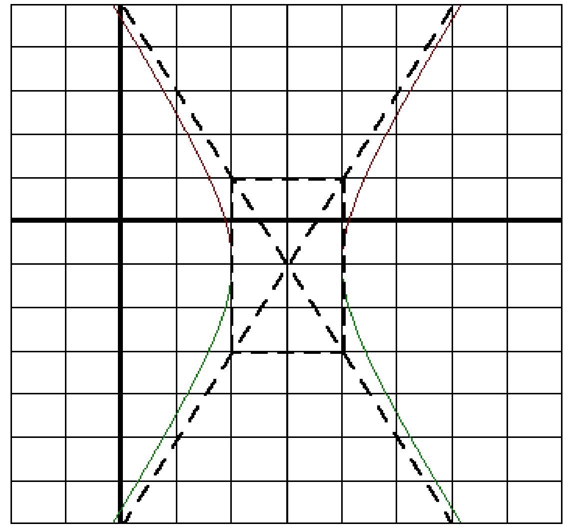
(b) $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Asymptotes: $y - k = \pm \frac{a}{b}(x - h)$

Vertices: $(h,\ k \pm a)$, Foci: $(h,\ k \pm c)$

Once again in both of these cases the length of the major axis is 2a. The length of the minor axis is 2b. To find the c value in any of these graphs, use the equation $c = \sqrt{a^2 + b^2}$. The eccentricity is still $e = \frac{c}{a}$.

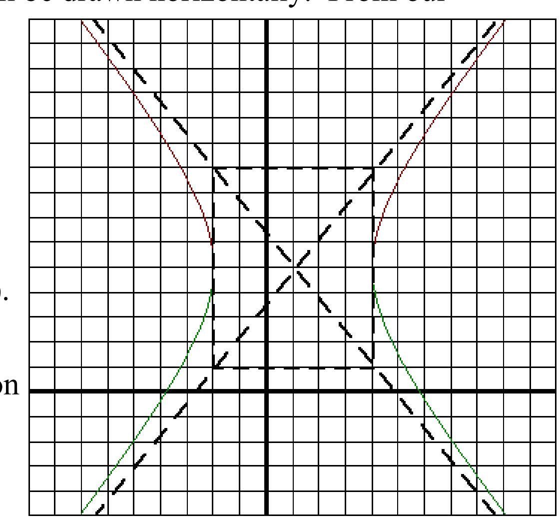
EXAMPLE: Graph $\frac{(x-3)^2}{1}-\frac{(y+1)^2}{4}=1$ and identify the foci, eccentricity, center, length of the transverse and conjugate axis, vertices, and the equations of the asymptotes.



The x comes first, so we know this hyperbole will be drawn horizontally. Since a always comes first we know that a = 1 and b = 2. We can find c by the formula $c=\sqrt{a^2+b^2}$: $c=\sqrt{1^2+2^2}$ So $c=\sqrt{5}\approx 2.24$. We start at the center, which is (3, -1). Remember to take the opposite sign of what is given. Since this is drawn horizontally I will add 1 and subtract 1 from the x value of the center since a is 1. The a always goes in the direction the hyperbola is opening up. This will give us the vertices (2, -1) and (4, -1). Then I will go up and down 2 from the y value of our vertex since this is our b. The foci run along the same line as the transverse axis, so from the x value of the center I will add 2.24 and subtract 2.24 to find the foci. We can write the foci as $(3\pm\sqrt{5},\ -1)$. We know the length of the transverse axis is 2(1) = 2. The length of the conjugate axis is 2(2) = 4. Then we can find the eccentricity, which is $e=\frac{2.24}{1}=2.24$. The equation of the asymptotes is $y+1=\pm 2(x-3)$.

EXAMPLE: Graph $16x^2-9y^2-32x+90y-353=0$ and identify the foci, eccentricity, center, length of the transverse and conjugate axis, vertices, and the equations of the asymptotes.

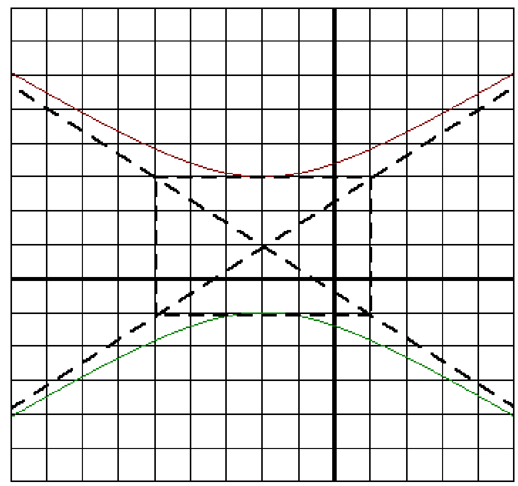
First we need to get this into the correct form. The first thing to do is subtract the 1 from both sides and group the x and y terms together: $16x^2-32x-9y^2+90y=353$. Now I will factor the two x terms and the two y terms separately: $16(x^2-2x)-9(y^2-10y)=353$. Now it is time to complete the square. In the first term we will divide the -2 by 2 and square it. You will get 1. We will add 1 to the left hand side however we do NOT add 1 to the right side. The 1 is really being multiplied by the 16, so we will add 1(16) = 16 to both sides. In the second term we will divide the -10 by 2 and square it. You will get 25. We will add 25 to the left hand side however we do NOT add 25 to the right side. The 25 is really being multiplied by the -9, so we will add -9(25) = -225 to both sides: $16(x^2-2x+1)-9(y^2-10y+25)=353+16+-225$. We can simplify the right side and factor the left side: $16(x-1)^2-9(y-5)^2=144$. Now divide both sides by 144 and reduce to get: $\frac{(x-1)^2}{9}-\frac{(y-5)^2}{16}=1$. The x comes first, so we know this hyperbola will be drawn horizontally. From our



From our equation we know that a is 3 and b is 4. We can find c by the formula $c=\sqrt{a^2+b^2}$: $c=\sqrt{3^2+4^2}$. So $c=\sqrt{25}=5$. We start at the center, which is (1, 5). Since this is drawn horizontally I will add 3 and subtract 3 from the x value of the center since a is 3. This will gives us the vertices (-2, 5) and (4, 5). Then I will go up and down 4 since this is our b. To find the foci I will add 5 and subtract 5 from the x value of the center. The foci can be written as: (6, 5) & (-4, 5). The length of the transverse axis is 2(3) = 6. The length of the conjugate axis is 2(4) = 8. The eccentricity is $e=\frac{5}{3}\approx 1.67$. The equation of the asymptotes is: $y-5=\pm\frac{4}{3}(x-1)$.

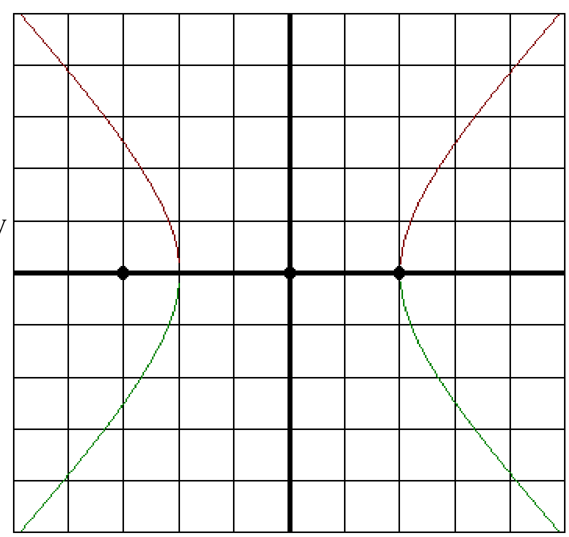
EXAMPLE: Graph $9y^2 - 18y - 4x^2 - 16x - 43 = 0$ and identify the foci, eccentricity, center, length of the transverse and conjugate axis, vertices, and the equations of the asymptotes.

First we need to get this into the correct form. The first thing to do is subtract the 1 from both sides and group the x and y terms together: $9y^2 - 18y - 4x^2 - 16x = 43$. Now I will factor the two x terms and the two y terms separately: $9(y^2 - 2x) - 4(x^2 + 4x) = 43$. Now it is time to complete the square. In the first term we will divide the -2 by 2 and square it. You will get 1. We will add 1 to the left hand side however we do NOT add 1 to the right side. The 1 is really being multiplied by the 9, so we will add 1(9) = 16 to both sides. In the second term we will divide the 4 by 2 and square it. You will get 4. We will add 4 to the left hand side however we do NOT add 4 to the right side. The 4 is really being multiplied by the -4, so we will add -4(4) = -16 to both sides: $9(y^2 - 2x + 1) - 4(x^2 + 4x + 4) = 43 + 9 + -16$. We can simplify the right side and factor the left side: $9(y-1)^2 - 4(x+2)^2 = 36$. Now divide both sides by 36 and reduce to get: $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$. The y comes first, so we know this hyperbola will be drawn vertically. From our From our equation we know that a is 2 and b is 3. We can find c by the formula $c = \sqrt{a^2 + b^2}$: $c = \sqrt{2^2 + 3^2}$. So $c = \sqrt{13} \approx 3.61$. We start at the center, which is (1, -2). Since this is drawn vertically I will add 2 and subtract 2 from the y value of the center since a is 2. This will gives us the vertices (-2, 3) and (-2, -1). Then I will go left and right 3 since this is our b. To find the foci I will add 3.61 and subtract 3.61 from the y value of the center. The foci can be written as: $(-2,\ 1 \pm \sqrt{13})$. The length of the transverse axis is 2(2) = 4. The length of the conjugate axis is 2(3) = 6. The eccentricity is $e = \frac{\sqrt{13}}{2} \approx 1.8$. The equation of the asymptotes is: $y - 1 = \pm\frac{2}{3}(x+2)$.

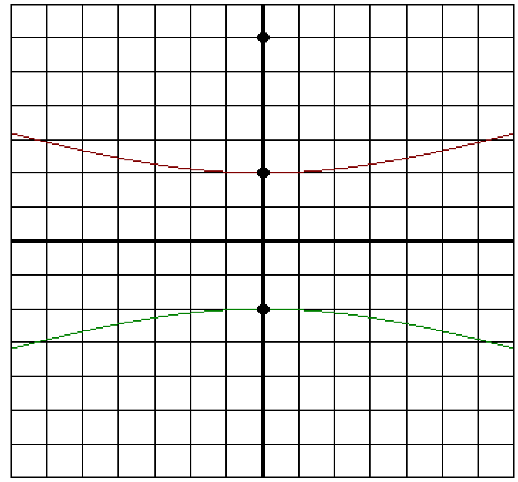


EXAMPLE: Find an equation for the hyperbola centered at the origin with focus at (-3, 0) and vertex at (2, 0).

We know that c = 3 because this is how far the focus is from (0, 0). We also know a = 2 since that is how far the vertex is from the center. We can use $c = \sqrt{a^2 + b^2}$ to find b^2: $3 = \sqrt{2^2 + b^2}$. After squaring both sides we get: $9 = 4 + b^2$. Solving for b^2 we get $b^2 = 5$. Now we have everything we need to find our equation. From our graph we know this hyperbola is drawn horizontally, so it must use the formula $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Plugging in our known variables will give us: $\frac{x^2}{2^2} - \frac{y^2}{5} = 1$.

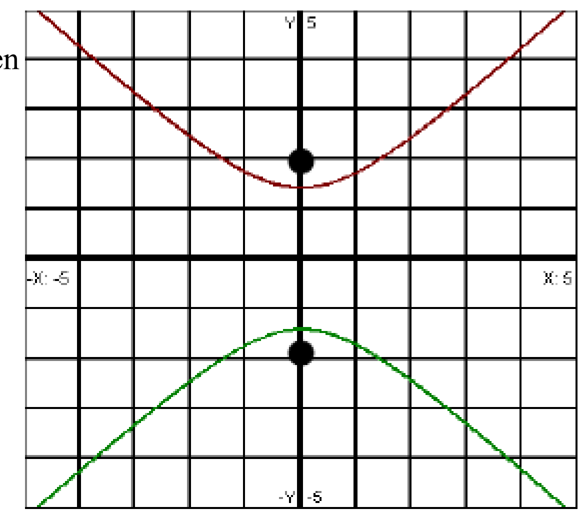


EXAMPLE: Find an equation for the hyperbola with a focus at (0, 6) and vertex at $(0, \pm 2)$.



First let's plot the above points. From this we can conclude that the hyperbola must be centered at (0, 0) since this point is halfway between our vertices. We also know that c = 6 because this is how far the focus is from the center. We also know a = 2 since that is how far the vertices are from the center. We can use $c = \sqrt{a^2 + b^2}$ to find b^2: $6 = \sqrt{2^2 + b^2}$. After squaring both sides we get: $36 = 4 + b^2$. Solving for b^2 we get $b^2 = 32$. Now we have everything we need to find our equation. From our graph we know this hyperbola is drawn vertically, so it must use the formula $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Plugging in our known variables will give us: $\frac{y^2}{2^2} - \frac{x^2}{32} = 1$.

EXAMPLE: Find an equation for the hyperbola with foci at $(0, \pm 2)$ and with an asymptote of $y = x$.



First let's plot the above points. From this we can conclude that the hyperbola must be centered at (0, 0) since this point is halfway between our foci. We also know that c = 2 because this is how far the focus is from the center. The slope of the asymptote is 1. From plotting our points we know that this hyperbola is drawn vertically. This means that the equation of the asymptote is $y = \pm \frac{a}{b} x$. So we know that $\frac{a}{b} = 1$. This tells us a = b. Since the foci is at $(0, \pm 2)$, we know that c = 2. We will use the formula $c = \sqrt{a^2 + b^2}$. Since we know that a = b, we can substitute an a for b. We can also put in a 2 for c: $2 = \sqrt{a^2 + a^2}$. This simplifies to: $2 = \sqrt{2a^2}$. Now square both sides to get: $4 = 2a^2$. So we know that $a^2 = 2$. Since the foci are drawn vertically, we know the hyperbola opens up and down. Therefore we use the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Our equation is $\frac{y^2}{2} - \frac{x^2}{2} = 1$.