7.2 The Inverse Trigonometric Functions (Continued)

When finding the inverse secant or inverse cosecant functions, we can use the below formulas. There is no equivalent formula for the inverse cotangent function, so it's definition is a little different.

$y = \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$ where $|x| \geq 1$ and $0 \leq y \leq \pi$ and $y \neq \frac{\pi}{2}$

$y = \csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$ where $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $y \neq 0$

$y = \cot^{-1} x$ where $-\infty < x < \infty$ and $0 \leq y \leq \pi$

EXAMPLE: Find the exact value of $\sec^{-1}(2)$.

First we will use the formula: $\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right)$. Since $0 \leq \cos^{-1}\left(\frac{1}{x}\right) \leq \pi$, this means we need to find an answer in either the first or second quadrant. Since the number inside the parenthesis is positive, we will look in the first quadrant. Using the unit circle, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

EXAMPLE: Find the exact value of $\csc^{-1}\left(-\sqrt{2}\right)$.

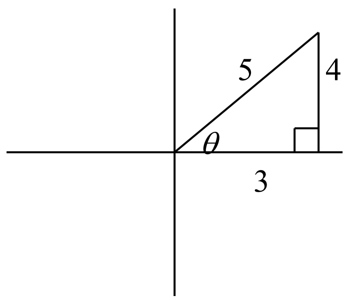
First we will use the formula: $\csc^{-1}\left(-\sqrt{2}\right) = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. Since $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{1}{x}\right) \leq \frac{\pi}{2}$, this means we need to find an answer in either the first or fourth quadrant. Since the number inside the parenthesis is negative, we will look in the fourth quadrant. Therefore, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{7\pi}{4}$ using the unit circle. However, we need to make sure our answer is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Therefore, we will subtract 2π from our answer: $\frac{7\pi}{4} - 2\pi = -\frac{\pi}{4}$.

EXAMPLE: Find the exact value of $\cot^{-1}\left(-\sqrt{3}\right)$.

Since $0 \leq \cot^{-1}(x) \leq \pi$, this means we need to find an angle in either the first or second quadrant whose cotangent equals $-\sqrt{3}$. We can use the unit circle for this one. Remember that cotangent is the same as the x value divided by the y value on the unit circle. quadrant. Therefore, $\cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$.

EXAMPLE: Use a sketch to find the exact value: $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$.

These problems involve drawing a triangle and labeling the sides like we did in a previous section. The inverse trig function will tell you where to draw the triangle. In our example there is an inverse cosine. The inverse cosine's range will tell us where we can draw the triangle. From the last section, the range for the inverse cosine is $0 \le y \le \pi$. This corresponds to the first and second quadrant. Since the fraction $\frac{3}{5}$ is positive, the only quadrant the triangle can be drawn in is the first quadrant. We know that the adjacent side is 3 and the hypotenuse is 5. The Pythagorean Theorem will give us the opposite side, which is 4.

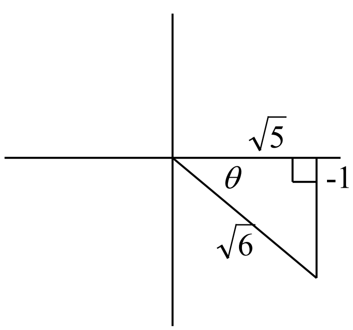


The sine on the outside of our problem tells us how to write our answer. From our drawing, sine is 4 over 5, so we write our answer as:

$$\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \frac{4}{5}$$

EXAMPLE: Use a sketch to find the exact value: $\cos\left(\sin^{-1}\left(-\frac{1}{\sqrt{6}}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in this case we have an inverse sine. The inverse sine's range will tell us where we can draw the triangle. From the last section, the range for the inverse sine is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. This corresponds to the first and fourth quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the fourth quadrant. We know that the opposite side is -1 and the hypotenuse is $\sqrt{6}$. The Pythagorean Theorem will give us the adjacent side, which is $\sqrt{5}$.

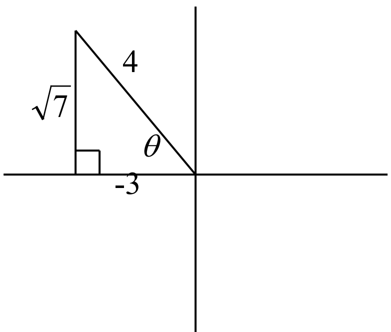


The cosine on the outside of our problem tells us how to write our answer. From our drawing, cosine is $\sqrt{5}$ over $\sqrt{6}$, so we write our answer as:

$$\cos\left(\sin^{-1}\left(-\frac{1}{\sqrt{6}}\right)\right) = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

EXAMPLE: Use a sketch to find the exact value: $\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in our case there is an inverse cosine. The inverse cosine's range will tell us where we can draw the triangle. From the last section, the range for the inverse cosine is $0 \leq y \leq \pi$. This corresponds to the first and second quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the second quadrant. We know that the adjacent side is -3 and the hypotenuse is 4. The Pythagorean Theorem will give us the opposite side: $\sqrt{7}$.

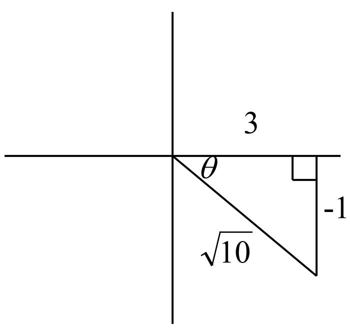


The tangent on the outside of our problem tells us how to write our answer. From our drawing, tangent is $\sqrt{7}$ over -3, so:

$$\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right) = -\frac{\sqrt{7}}{3}$$

EXAMPLE: Find the exact value: $\csc\left(\tan^{-1}\left(-\frac{1}{3}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in this case we have an inverse tangent. The inverse tangent's range will tell us where we can draw the triangle. From the last section, the range for the inverse tangent is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. This corresponds to the first and fourth quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the fourth quadrant. We know that the opposite side is -1 and the adjacent is 3. The Pythagorean Theorem will give us the adjacent side: $\sqrt{10}$.



The cosecant on the outside of our problem tells us how to write our answer. From our drawing, cosecant is $\sqrt{10}$ over -1, so we write your answer as:

$$\csc\left(\tan^{-1}\left(-\frac{1}{3}\right)\right) = \frac{\sqrt{10}}{-1} = -\sqrt{10}.$$

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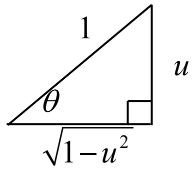
EXAMPLE: Use right triangles to write in algebraic form: $\cos\left(\sin^{-1} u\right)$. Assume that u is positive and that the given inverse trigonometric function is defined for the expression in u.

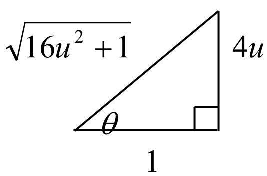
These problems involve drawing a triangle and labeling the sides with algebraic expressions. In this problem we are told that u is positive, so the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\cos\left(\sin^{-1}\frac{u}{1}\right)$ We know that the opposite side is u and the hypotenuse is 1. We can use the Pythagorean theorem to find the hypotenuse: $1^2 = u^2 + b^2 \Rightarrow b^2 = 1 - u^2$ So we have $b = \sqrt{1-u^2}$



The cosine on the outside of our problem tells us how to write our answer. From our drawing, cosine is $\sqrt{1-u^2}$ over 1 so we write our answer as: $\cos\left(\sin^{-1} u\right) = \sqrt{1-u^2}$.

EXAMPLE: Use right triangles to write in algebraic form: $\sec\left(\tan^{-1} 4u\right)$. Assume that u is positive and that the given inverse trigonometric function is defined for the expression in u.

Again we will draw a triangle for this one. In this problem we are told that u is positive, so the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\tan^{-1}\frac{4u}{1}\right)$ We know that the adjacent side is 1 and the opposite side is $4u$. We can use the Pythagorean theorem to find the hypotenuse: $c^2 = (4u)^2 + (1)^2$. So we have $c = \sqrt{16u^2+1}$



The secant on the outside of our problem tells us how to write our answer. From our drawing, secant is $\sqrt{16u^2+1}$ over 1 so we write our answer as: $\sec\left(\tan^{-1} 4u\right) = \sqrt{16u^2+1}$