7.3 Trigonometric Equations

You should have a unit circle sheet. If not, this is available on the website. This allows us to see the exact values of certain angles between 0 and 360 degrees. Now we don't need to use reference angles. This section will cover how to solve trigonometric equations which is one skill you will need in calculus. The main strategy is to isolate the trig function. Then we will take the inverse trig function of both sides to get the answer.

EXAMPLE: Solve for x: $\cos x = \frac{\sqrt{3}}{2}$ on $[0,\ 360^\circ)$.

The cosine is already isolated, so now we will take the inverse cosine of both sides.

$\cos^{-1}(\cos x) = \cos^{-1}\frac{\sqrt{3}}{2}$. Now simplify.

$x = \cos^{-1}\frac{\sqrt{3}}{2}$ What we need to do now is look at the unit circle sheet and find ANY angles between 0° and 360° that give an x value of $\frac{\sqrt{3}}{2}$. Remember x corresponds to cosine and y corresponds to sine.

$x = 30^\circ,\ 330^\circ$ Both of these will give a value of $\frac{\sqrt{3}}{2}$.

EXAMPLE: Solve for x: $-2\sin x = 1$.

$\sin x = -\frac{1}{2}$ First we isolated the sine. Now we need to take the inverse sine of both sides.

$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{1}{2}\right)$ Simplify.

$x = \sin^{-1}\left(-\frac{1}{2}\right)$ We need to find ANY angles on the unit circle that give a y value of $-\frac{1}{2}$.

$x = \frac{7\pi}{6},\ \frac{11\pi}{6}$

The above is not my complete answer. This problem did not give an interval like $[0,\ 2\pi]$ to find your answers. Because of this, our answers will not only be in the first revolution of the circle. If no interval is given we need to add a $2\pi k$ to our answer. The k value represents how many times we are going around the circle until we come to our answers. So we will write: $x = \frac{7\pi}{6} + 2\pi k,\ \frac{11\pi}{6} + 2\pi k$. We could have also written our answers in degrees as well: $x = 210^\circ + 360^\circ k,\ 330^\circ + 360^\circ k$.

EXAMPLE: Solve for x: $2\cos x + \sqrt{2} = 0$ on $[0,\ 2\pi)$.

$\cos x = -\dfrac{\sqrt{2}}{2}$ First we isolate the cosine. Now we need to take the inverse cosine of both sides.

$\cos^{-1}(\cos x) = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ Simplify.

$x = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ We need to find ANY angles on the unit circle that give a x value of $-\dfrac{\sqrt{2}}{2}$.

$x = \dfrac{3\pi}{4},\ \dfrac{5\pi}{4}$ These are my only answer since they gave us an interval, $[0,\ 2\pi]$. Since this interval is given in radians we must write our answers in terms of radians.

EXAMPLE: Solve for x: $\cos^2 x = \dfrac{3}{4}$ on $[0,\ 2\pi\ )$.

To solve this one, take the square root of both sides: $\cos x = \pm\sqrt{\dfrac{3}{4}}$. You can take the square root of the top and bottom separately: $\cos x = \pm\dfrac{\sqrt{3}}{2}$. To solve this, look on the unit circle and find all angles that have an x value of $\pm\dfrac{\sqrt{3}}{2}$. The angles are: $\dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$.

EXAMPLE: Solve for x: $5\csc x - 3 = 2$ on $[0,\ 2\pi)$.

First isolate the cosecant by adding 3 to both sides and then dividing both sides by 5.

$\csc x = 1$ We will use the identity $\csc x = \dfrac{1}{\sin x}$. So now the problem becomes:

$\dfrac{1}{\sin x} = \dfrac{1}{1}$ Cross multiply.

$\sin x = 1$ Take the inverse sine of both sides.

$\sin^{-1}(\sin x) = \sin^{-1} 1$ Simplify.

$x = \sin^{-1} 1$ Look on the unit circle and find ANY angles that give a y value of 1.

$x = \dfrac{\pi}{2}$ This is the only single value on the unit circle.

EXAMPLE: Solve for θ: $\sqrt{3}\cot\theta + 1 = 0$ on $[0,\ 360^\circ)$

First isolate the cotangent by subtracting one from both sides and then dividing both sides by $\sqrt{3}$.

$\cot\theta = -\dfrac{1}{\sqrt{3}}$ — Now use the formula: $\dfrac{1}{\tan\theta} = \cot\theta$.

$\dfrac{1}{\tan\theta} = -\dfrac{1}{\sqrt{3}}$ — Cross multiply.

$\tan\theta = -\sqrt{3}$ — Since there is no tangent on our unit circle, look for ANY angle such that if you divide the y by x, (y/x) you will get $-\sqrt{3}$.

$\theta = 120^\circ,\ 300^\circ$

EXAMPLE: Solve for θ: $\cos 2\theta = \dfrac{1}{2}$.

The cosine is isolated, so now we will take the inverse cosine of both sides:

$\cos^{-1}(\cos 2\theta) = \cos^{-1}\dfrac{1}{2}$ — Simplify.

$2\theta = \cos^{-1}\dfrac{1}{2}$ — We need to find ANY angles on the unit circle that give a x value of $\dfrac{1}{2}$. Since there is no interval given, we need to add a 360k to our answers.

$2\theta = 60^\circ + 360^\circ k$
$2\theta = 300^\circ + 360^\circ k$ — For each of our answers we need to solve for θ by dividing by 2.

$\theta = 30^\circ + 180^\circ k$
$\theta = 150^\circ + 180^\circ k$ — These are our answers.

EXAMPLE: Solve for θ: $\sin(2\theta) = -\dfrac{\sqrt{3}}{2}$ on $[0,\ 2\pi)$.

We will proceed the same way we did the previous example. In this one we need to take the inverse sine of both sides:

$\sin^{-1}(\sin 2\theta) = \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$2\theta = \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ — We need to find ANY angles on the unit circle that give a x value of $-\dfrac{\sqrt{3}}{2}$. This will be $\dfrac{4\pi}{3}$ and $\dfrac{5\pi}{3}$. On this one, because there is something inside the trig function that is not just theta, we will also be using the k values. Because we need to use radians, we will add $2\pi k$.

$2\theta = \frac{4\pi}{3} + 2\pi k$, $2\theta = \frac{5\pi}{3} + 2\pi k$ For each of our answers we need to solve for θ by dividing both sides by 2.

$\theta = \frac{2\pi}{3} + \pi k$, $\theta = \frac{5\pi}{6} + \pi k$ These are our general solutions. To get the answers in our interval, we will be putting values in for k. First we will start k = 0, 1, 2, …until we get a number that is outside our interval.

$\theta = \frac{2\pi}{3} + \pi k$ When k = 0, we have: $\theta = \frac{2\pi}{3} + \pi(0)$, so $\theta = \frac{2\pi}{3}$

When k = 1, we have: $\theta = \frac{2\pi}{3} + \pi(1)$, so $\theta = \frac{5\pi}{3}$

If we let k = 2, then we get something that is more than 2π, which is outside our interval, so we will stop.

$\theta = \frac{5\pi}{6} + \pi k$ When k = 0, we have: $\theta = \frac{5\pi}{6} + \pi(0)$, so $\theta = \frac{5\pi}{6}$

When k = 1, we have: $\theta = \frac{5\pi}{6} + \pi(1)$, so $\theta = \frac{11\pi}{6}$

If we let k = 2, then we get something that is more than 2π, which is outside our interval, so we will stop.

So our answers to this problem are: $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}, \frac{11\pi}{6}$.

EXAMPLE: Solve for θ: $2\sin\left(\frac{\theta}{2}\right) - 1 = 0$ on $[0, 2\pi)$.

First we need to solve the above for sine: $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}$. We will proceed the same way we did in the previous example. In this one we need to take the inverse sine of both sides. For this problem I will do the whole problem in degrees and then convert my final answers. This may be an easier approach for you.

$$\sin^{-1}\left(\sin\left(\frac{\theta}{2}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right)$$

$\frac{\theta}{2} = \sin^{-1}\left(\frac{1}{2}\right)$ We need to find ANY angles on the unit circle that give an y value of $\frac{1}{2}$. This will be 30 and 150 degrees. Now we set up our two equations as before.

$\frac{\theta}{2} = 30^\circ + 360^\circ k$, $\frac{\theta}{2} = 150^\circ + 360^\circ k$ For each of our answers we multiply both sides by 2.

$\theta = 60^\circ + 720^\circ k$, $\theta = 300^\circ + 720^\circ k$ These are our general solutions. To get the answers in our interval, we will be putting values in for k. First we start with k = 0, 1, 2, … until we get an angle that is outside our interval.

$\theta = 60^\circ + 720^\circ k$ When k = 0, we have $\theta = 60^\circ + 720^\circ(0)$, so $\theta = 60^\circ = \frac{\pi}{3}$.

If we let k = 1 then we get an angle that is more than 360 degrees, which is outside our interval, so we will stop.

$\theta = 300^\circ + 720^\circ k$ When k = 0, we have $\theta = 300^\circ + 720^\circ(0)$, so $\theta = 300^\circ = \frac{5\pi}{3}$.

If we let k = 1 then we get an angle that is more than 360 degrees, which is outside our interval, so we will stop.

Therefore, our solutions to this problem are: $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

EXAMPLE: Solve the equation: $\cos^2 x - \cos x = 0$ on $[0,\ 2\pi)$.

For this one the only thing we can do is factor out the common factor, which is $\cos x$: $\cos x(\cos x - 1) = 0$. Now we need to set each factor equal to zero. We will get $\cos x = 0$ and $\cos x - 1 = 0$. We need to solve each equation separately. For the equation $\cos x = 0$ we need to look at the unit circle where the x value is zero. This will happen at $\frac{\pi}{2}$ and at $\frac{3\pi}{2}$. For the second equation $\cos x - 1 = 0$ this is the same as $\cos x = 1$. Again we look at the unit circle and find all places where the x value is 1. We will get 0. The angle of zero radians is also the same as 2π radians, but we don't write the 2π because that is not included on our interval.

EXAMPLE: Solve the equation: $2\cos^2 x + \cos x - 1 = 0$ on $[0,\ 2\pi)$.

We can factor this one: $(2\cos x - 1)(\cos x + 1) = 0$. Now set each part equal to zero. We get $2\cos x - 1 = 0$ and $\cos x + 1 = 0$. Solving the first equation we will get $\cos x = \frac{1}{2}$. We look on the unit circle and look for any x values that are $\frac{1}{2}$. This will happen at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Solving the second equation you will get $\cos x = -1$. The angle that gives an x value of negative one is π. Therefore our answers are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

EXAMPLE: Solve the equation: $\sin x \cos^2 x = 2\sin x$ on $[0,\ 360^\circ)$.

We need to set this equal to zero: $\sin x \cos^2 x - 2\sin x = 0$. Now factor out a common factor of $\sin x$: $\sin x(\cos^2 x - 2) = 0$. Now set each factor equal to zero. We have $\sin x = 0$. Looking at our unit circle we see that x is 0° and 180°. The other equation gives us $\cos^2 x = 2$. Taking the square root we get $\cos x = \pm\sqrt{2}$. This means that $\cos x = \pm 1.41$. Since this number is larger than one, this will not give us any solutions because of the domain of the cosine. So our answers for x are 0° and 180°.

EXAMPLE: Solve the equation: $\tan^3 x - \tan^2 x - 3\tan x + 3 = 0$ on $[0, 2\pi)$.

This one can be factored by grouping. I will factor out a $\tan^2 x$ from the first two terms and then a negative 3 from the second two terms. This will give us: $\tan^2 x(\tan x - 1) - 3(\tan x - 1) = 0$. There is a common factor of $\tan x - 1$ that I will factor out: $(\tan x - 1)(\tan^2 x - 3) = 0$. Now we set both factors individually equal to zero. For the first equation we have $\tan x - 1 = 0$, in which $\tan x = 1$, so $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ from the unit circle. For the second equation we have $\tan^2 x - 3 = 0$, in which $\tan x = \pm\sqrt{3}$. Since we get a positive or a negative this means that we will get an answer in all four quadrants of the unit circle. We get the following: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. Our final answer if all 6 of the following angles: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{4}, \frac{5\pi}{4}$.

EXAMPLE: Solve the equation: $2\sin^2\theta - \cos 2\theta = 0$ on $[0, 360^\circ)$.

Since there are both sines and cosines we need to use an identity to get all the terms to have the same trig value. Since I notice there is already a sine in the problem I want to use $\cos 2\theta = 1 - 2\sin^2\theta$. So now the problem is: $2\sin^2\theta - (1 - 2\sin^2\theta) = 0$. Simplifying we get: $4\sin^2\theta - 1 = 0$. When we solve this we get $\sin^2\theta = \pm\frac{1}{4}$ so $\sin\theta = \pm\frac{1}{2}$. Then values off the unit circle are: $30^\circ, 150^\circ, 210^\circ, 330^\circ$. Notice that since our interval was given in degrees we can write our answers in degrees.

EXAMPLE: Solve the equation: $\sin 2\theta = \cos\theta$ on $[0, 2\pi)$.

We need to use another identity on this one. This time we will use $\sin 2\theta = 2\sin\theta\cos\theta$. So now our problem becomes: $2\sin\theta\cos\theta = \cos\theta$. Setting it equal to zero will give us: $2\sin\theta\cos\theta - \cos\theta = 0$. We can factor out a cosine to get: $\cos\theta(2\sin\theta - 1) = 0$. Setting the first term equal to zero will give us $\cos\theta = 0$, so we know from the unit circle $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. Setting the second term equal to zero we will get $2\cos\theta - 1 = 0$, so $\sin\theta = \frac{1}{2}$. Then we know from the unit circle $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. So our final answer is: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

EXAMPLE: Solve the equation: $2\cot^2 x + \csc^2 x - 2 = 0$ on $[0, 360^\circ)$.

Once again we need to use a trig identity so that all of these are in terms of the same trig value. We can either use all cotangents or all cosecants. I will use the identity $\cot^2 x = \csc^2 x - 1$: $2(\csc^2 x - 1) + \csc^2 x - 2 = 0$. Simplifying we get: $3\csc^2 x - 4 = 0$. Solving for cosecant we get $\csc^2 x = \frac{4}{3}$. I will now use the identity $\csc^2 x = \frac{1}{\sin^2 x}$. This will change the problem into: $\frac{1}{\sin^2 x} = \frac{4}{3}$. Cross multiplying gives us $4\sin^2 x = 3$, so $\sin^2 x = \frac{3}{4}$. Solving for sine we get $\sin x = \pm\frac{\sqrt{3}}{2}$. Then we know from the unit circle:

$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

EXAMPLE: Solve the equation: $\tan 2\theta - 2\cos\theta = 0$ on $[0,\ 360^\circ)$.

Below you will see each step towards solving this problem:

$\frac{\sin 2\theta}{\cos 2\theta} - 2\cos\theta = 0$ — First we use an identity for the tangent term.

$\frac{\sin 2\theta}{\cos 2\theta} - \frac{2\cos\theta\cos 2\theta}{\cos 2\theta} = 0$ — Get common denominators.

$\frac{\sin 2\theta - 2\cos\theta\cos 2\theta}{\cos 2\theta} = 0$ — Add common fractions.

$\sin 2\theta - 2\cos\theta\cos 2\theta = 0$ — Cross multiply. We want to factor this, but nothing to factor in this form.

$2\sin\theta\cos\theta - 2\cos\theta\cos 2\theta = 0$ — We can apply an identity to the $\sin 2\theta$.

$2\cos\theta(\sin\theta - \cos 2\theta) = 0$ — Factor out the common factor. Now we need to use an identity for $\cos 2\theta$. We have three to choose from, but since there is a sine inside the Parenthesis, choose the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$2\cos\theta(\sin\theta - (1 - 2\sin^2\theta)) = 0$ — Apply the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$2\cos\theta(\sin\theta - 1 + 2\sin^2\theta) = 0$ — Distribute the minus sign.

$2\cos\theta(2\sin^2\theta + \sin\theta - 1) = 0$ — Rearrange.

$2\cos\theta(2\sin\theta - 1)(\sin\theta + 1) = 0$ — Factor. Now set each equal to zero and solve:

$2\cos\theta = 0$	$2\sin\theta - 1 = 0$	$\sin\theta + 1 = 0$
$\cos\theta = 0$	$\sin\theta = \frac{1}{2}$	$\sin\theta = -1$
$\theta = 90^\circ,\ 270^\circ$	$\theta = 30^\circ,\ 150^\circ$	$\theta = 270^\circ$

Therefore our answers are: $\theta = 30^\circ,\ 90^\circ,\ 150^\circ,\ 270^\circ$