

7.4 Trigonometric Identities

This section will help you practice your trigonometric identities. We are going to establish an identity. What this means is to work out the problem and show that both sides of the identity are the same. First let's look at a list of identities. Most of these are derived directly from the unit circle.

List of Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \quad \csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \sin^2 \theta = 1 - \cos^2 \theta \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$\sec^2 \theta = 1 + \tan^2 \theta \quad \tan^2 \theta = \sec^2 \theta - 1 \quad \csc^2 \theta = 1 + \cot^2 \theta \quad \cot^2 \theta = \csc^2 \theta - 1$$

When working out these identities, you can try on or more of the following techniques. I will explain each technique with examples:

- 1.) Change everything into sines and cosines.
- 2.) Use factoring to simplify the expression if possible.
- 3.) Get common denominators if there are fractions.
- 4.) Multiply both sides by a conjugate.

Of course, as we use the above techniques, be sure to refer back to the list of identities I gave you above. You might need to use some of them to simplify. One you will see come up often is $\sin^2 \theta + \cos^2 \theta = 1$.

EXAMPLE: Establish the identity: $\csc \theta \cdot \tan \theta = \sec \theta$.

You want to show that one side of the equation equals the other side. In these problems you are NOT allowed to do operations like adding or subtracting things from one side to the other. Think of each side as independent. We are not going to do anything with the right hand side. On the left side we will use the first technique and change the cosecant and tangent functions into sines and cosines:

$$\frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta} = \sec \theta$$

We can now cancel the sines from the left side of the equation.

$$\frac{1}{\cos \theta} = \sec \theta$$

We can change the fraction on the left side into secant.

$$\sec \theta = \sec \theta$$

Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\frac{\sin^4 \theta - \cos^4 \theta}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta)$.

Another technique I mentioned is factoring. We can factor the top because of difference of squares.

$$\frac{(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta)}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta) \quad \text{We know } \sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{(\sin^2 \theta - \cos^2 \theta)}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta) \quad \text{We can factor the top again by difference of squares.}$$

$$\frac{(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta) \quad \text{We want to factor a negative out of the first term.}$$

$$\frac{-(-\sin \theta + \cos \theta)(\sin \theta + \cos \theta)}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta) \quad \text{Now switch the order in the first term on top.}$$

$$\frac{-(\cos \theta - \sin \theta)(\sin \theta + \cos \theta)}{\cos \theta - \sin \theta} = -(\cos \theta + \sin \theta) \quad \text{Now we can cancel the } \cos \theta - \sin \theta \text{ terms.}$$

$$-(\sin \theta + \cos \theta) = -(\cos \theta + \sin \theta) \quad \text{Both sides are equal so the proof is done.}$$

EXAMPLE: Establish the identity: $\frac{\cos x - 2 \sin x \cos x}{\cos^2 x - \sin^2 x + \sin x - 1} = \cot x$.

$$\frac{\cos x(1 - 2 \sin x)}{\cos^2 x - \sin^2 x + \sin x - 1} = \cot x \quad \text{First we can factor the numerator. Now we want to get all sines on the bottom. We can use the identity } \cos^2 x = 1 - \sin^2 x.$$

$$\frac{\cos x(1 - 2 \sin x)}{(1 - \sin^2 x) - \sin^2 x + \sin x - 1} = \cot x \quad \text{Now simplify the denominator.}$$

$$\frac{\cos x(1 - 2 \sin x)}{-2 \sin^2 x + \sin x} = \cot x \quad \text{Factor the denominator.}$$

$$\frac{\cos x(1 - 2 \sin x)}{\sin x(-2 \sin x + 1)} = \cot x \quad \text{The part in parenthesis on top and bottom can be cancelled.}$$

$$\frac{\cos x}{\sin x} = \cot x \quad \text{We will use the identity } \cot x = \frac{\cos x}{\sin x}$$

$$\cot x = \cot x \quad \text{Both sides are equal so we are done.}$$

EXAMPLE: Establish the identity: $\cot \theta + \frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta$.

Since this problem has a fraction, I will follow technique #3, which says to get common denominators if there are fractions. At the same time I will also use the identity: $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

$$\frac{\cos \theta}{\sin \theta} \cdot \left(\frac{\cos \theta}{\cos \theta} \right) + \frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta$$

Now write as a single fraction.

$$\frac{\cos^2 \theta + 1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta$$

Now simplify the numerator.

$$\frac{1 - \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta$$

We will now use the identity $\sin^2 \theta = 1 - \cos^2 \theta$.

$$\frac{\sin^2 \theta}{\sin \theta \cos \theta} = \tan \theta$$

We can cancel a sine from the top and bottom.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta$$

We will use the identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

$$\tan \theta = \tan \theta$$

Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 2 \sec x$.

Once again we want to first get a single fraction so we need common denominators.

$$\frac{\cos x}{1 + \sin x} \cdot \left(\frac{\cos x}{\cos x} \right) + \frac{1 + \sin x}{\cos x} \cdot \left(\frac{1 + \sin x}{1 + \sin x} \right) = 2 \sec x$$

Now multiply and write as a single fraction.

$$\frac{\cos^2 x + (1 + \sin x)^2}{\cos x(1 + \sin x)} = 2 \sec x$$

We will expand the numerator.

$$\frac{\cos^2 x + \sin^2 x + 2 \sin x + 1}{\cos x(1 + \sin x)} = 2 \sec x$$

We will use the identity $\cos^2 x + \sin^2 x = 1$

$$\frac{1 + 2 \sin x + 1}{\cos x(1 + \sin x)} = 2 \sec x$$

Simplify the numerator.

$$\frac{2 \sin x + 2}{\cos x(1 + \sin x)} = 2 \sec x$$

Factor the numerator.

$$\frac{2(\sin x + 1)}{\cos x(1 + \sin x)} = 2 \sec x$$

We can cancel the $\sin x + 1$ from the top and bottom.

$$\frac{2}{\cos x} = 2 \sec x$$

We will use the identity $\sec x = \frac{1}{\cos x}$.

$$2 \sec x = 2 \sec x$$

Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\frac{\tan x + \cot x}{\sec x \csc x} = 1$.

We will use technique #1 and change everything into sines and cosines. This makes it easier to reduce.

$$\frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{1}{\cos x} \cdot \frac{1}{\sin x}} = 1$$

We need to get common denominators in the numerator.

$$\frac{\frac{\sin x}{\cos x} \cdot \left(\frac{\sin x}{\sin x}\right) + \frac{\cos x}{\sin x} \cdot \left(\frac{\cos x}{\cos x}\right)}{\frac{1}{\cos x} \cdot \frac{1}{\sin x}} = 1$$

Multiply and write as one fraction in the numerator.

$$\frac{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}{\frac{1}{\sin x \cos x}} = 1$$

Now use the identity $\sin^2 \theta + \cos^2 \theta = 1$.

$$\frac{\frac{1}{\sin x \cos x}}{\frac{1}{\sin x \cos x}} = 1$$

Flip over the bottom fraction and multiply.

$$\frac{1}{\sin x \cos x} \cdot \frac{\sin x \cos x}{1} = 1$$

We can cancel terms.

$$1 = 1$$

Both sides are the same so we are done.

EXAMPLE: Establish the identity: $\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 - \sin \theta}$.

We see that everything is already in terms of sine and cosine. Also notice that we can't factor and even though there are fractions, we don't need common denominators. The only other technique that we can use is technique #4, which says to multiply both sides by a conjugate. We can do this on either side. Please note that we are NOT allowed to cross multiply because we need to treat both sides separately.

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 - \sin \theta} \cdot \left(\frac{1 + \sin \theta}{1 + \sin \theta} \right)$$

I chose the right side, but we can choose either side to work with.

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta (1 + \sin \theta)}{1 - \sin^2 \theta}$$

Now use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta (1 + \sin \theta)}{\cos^2 \theta}$$

Now we can cancel the cosine from the top and bottom.

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{1 + \sin \theta}{\cos \theta}$$

Both sides are equal so we are done.

Let's do this problem again, but now let's work on the left side instead of the right side.

$$\left(\frac{1 - \sin \theta}{1 - \sin \theta} \right) \frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 - \sin \theta}$$

I chose the left side this time.

$$\frac{1 - \sin^2 \theta}{\cos \theta (1 - \sin \theta)} = \frac{\cos \theta}{1 - \sin \theta}$$

Now use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\frac{\cos^2 \theta}{\cos \theta (1 - \sin \theta)} = \frac{\cos \theta}{1 - \sin \theta}$$

Now we can cancel the cosine from the top and bottom.

$$\frac{\cos}{1 - \sin \theta} = \frac{\cos \theta}{1 - \sin \theta}$$

Both sides are equal so we are done.

Remember you only need to show that both sides are equal. What each side is does not matter, as long as both sides are the same. I am not looking for one exact way of doing these problems, because there may be more than one way to show that one side equals the other. Just make sure you logically show your steps. If you start with one statement and then jump down to the answer without showing how you got there, you will not receive full credit. Your answers to these problems will be these logical steps you show.