7.4 Trigonometric Identities

Factoring Review

This section and a future section will require factoring to simplify expressions. Let's look at a few review problems from precalculus, and then we will apply this to trigonometric expressions.

EXAMPLE: Factor $30x^3 + 42x^2$.

First, we need to find the greatest common factor (GCF) of the two terms. When we have different powers of *x*, the most we can factor out will be the lowest power of *x*. Therefore, part of our GCF is x^2. Next, we want to look at the numbers 30 and 42. We want to find the largest number that divides evenly into these two numbers. We see that both numbers are divisible by 6, and that would be the largest number that divides evenly into both. Putting this all together, the GCF of $30x^3$ and $42x^2$ is $6x^2$, so this is what we will factor out. First, we will write the following $30x^3 + 42x^2 = 6x^2\left(\quad \right)$. We need to find out what goes inside the parenthesis. To do this, divide each term by the GCF: $\dfrac{30x^3}{6x^2} = 5x$ and $\dfrac{42x^2}{6x^2} = 7$. These results go inside the parenthesis. So, $30x^3 + 42x^2 = 6x^2\left(5x + 7\right)$.

EXAMPLE: Factor $80\sin^4 x - 30\sin x$.

This problem is almost the same as the previous one, with the difference now is that we have a trig function instead of just an *x*. However, we will still factor this the same as with precalculus. We see that sine is common in both terms, so that will be part of our GCF. Next, we want to look at the numbers 80 and 30. We want to find the largest number that divides evenly into these two numbers. We see that both numbers are divisible by 10. Putting this all together, the GCF of $80\sin^4 x$ and $30\sin x$ is $10\sin x$, so this is what we will factor out. First, we will write the following $80\sin^4 x - 30\sin x = 10\sin x\left(\quad \right)$. We need to find out what goes inside the parenthesis. To do this, divide each term by the GCF: $\dfrac{80\sin^4 x}{10\sin x} = 8\sin^3 x$ and $\dfrac{-30\sin x}{10\sin x} = -3$. These results go inside the parenthesis. So, $80\sin^4 x - 30\sin x$ factors into $10\sin x\left(8\sin^3 x - 3\right)$.

EXAMPLE: Factor $x^2 + 13x + 36$ completely, if possible.

For these kinds of problems where the number in front of the leading *x* is 1, you are looking for two numbers that multiply to make the last number but add up to the middle number. All the terms are positive here, so when we write out our pairs of numbers, they will all be positive.

1.) First let's write all the pairs of numbers that multiply to make 36: 1,36 2,18 3,12 4,9 6,6
2.) From our list, we see that 4 and 9 will work since they multiply to make 36 but add together to make 13.
3.) We will write the factored expression as: $x^2 + 13x + 36 = \left(x + 4\right)\left(x + 9\right)$.

EXAMPLE: Factor $\cos^2 x - 13\cos x + 42$ completely. If the polynomial cannot be factored, say it is prime.

We notice here that the middle term is negative and last term is positive. The only way this will work is if both factors are negative. For example, –1 and –42. Then multiply to make a positive number but add to a negative number. So, for this problem when we list the pairs, then should both be negative.

1.) First let's write all the pairs of numbers that multiply to make 42: –1,–42 –2,–21 –3,–14 –6,–7
2.) From our list, we see that –6 and –7 will work since they multiply to make 42 but add together to make –13.
3.) When we write our answer, the only difference this time is that instead of using x's, we will replace that with cosine. We will write the factored expression as: $\cos x^2 - 13\cos x + 42 = (\cos x - 6)(\cos x - 7)$.

Trinomials with leading coefficients other than 1 require a different process to factor. We can factor by grouping by taking the middle term and turning it into two terms. This will turn the polynomial into one that has 4 terms which can then be factored by grouping.

Steps for factoring $ax^2 + bx + c$ by grouping (AC Method)

Multiply a times c. Then find two numbers that multiply to ac but add up to the middle term. From here you will factor by grouping as shown below.

EXAMPLE: Factor $6x^2 + 23x + 20$ using the grouping method, if possible.

1.) First, we will multiply $a \cdot c = 6 \cdot 20 = 120$.
2.) Now let's write all the pairs of numbers that multiply to make 120. These can all be positive since the middle term is positive:
 1,120 2,60 3,40 4,30 5,24 6,20 8,15 10,12
3.) We see that 8 and 15 will work since they multiply to make 120 but add together to make 23.
4.) We will rewrite the original expression as: $6x^2 + 8x + 15x + 20$. It does not matter which factor comes first.
5.) Now we need to factor this using the grouping method:

$$\begin{aligned} 6x^2 + 8x + 15x + 20 &= (6x^2 + 8x) + (15x + 20) \\ &= 2x(3x+4) + 5(3x+4) \\ &= (3x+4)(2x+5) \end{aligned}$$

EXAMPLE: Factor $2\tan^2 x + 9\tan x - 35$ using the grouping method, if possible.

1.) First, we will multiply $a \cdot c = 2(-35) = -70$.
2.) Now let's write all the pairs of numbers that multiply to make –70. Since the last term is negative, one of our factors will be positive and the other one will be negative. Below are all the factors and sign combinations:
 1, –70 –1, 70 2, –35 –2, 35 5, –14 –5, 14 7, –10 –7, 10
3.) We see that –5 and 14 will work since they multiply to make –70 but add together to make +9.
4.) We will rewrite the original expression as: $2\tan x^2 - 5\tan x + 14\tan x + 35$. It does not matter which factor comes first.
5.) Now we need to factor this using the grouping method:

$$\begin{aligned} 2\tan^2 x - 5\tan x + 14\tan x + 35 &= (2\tan^2 x - 5\tan x) + (14\tan x + 35) \\ &= \tan x(2\tan x - 5) + 7(2\tan x - 5) \\ &= (2\tan x - 5)(\tan x + 7) \end{aligned}$$

The rest of this section will help you practice your trigonometric identities. We are going to establish an identity. What this means is to work out the problem and show that both sides of the identity are the same. First let's look at a list of identities. Most of these are derived directly from the unit circle.

List of Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$ $\tan^2\theta = \sec^2\theta - 1$ $\csc^2\theta = 1 + \cot^2\theta$ $\cot^2\theta = \csc^2\theta - 1$

When working out these identities, you can try one or more of the following techniques (strategies). I will explain each technique with examples:

1.) Change everything into sines and cosines.

2.) Use factoring to simplify the expression if possible.

3.) Get common denominators if there are fractions.

4.) Multiply one side by a conjugate.

Of course, as we use the above techniques, be sure to refer back to the list of identities I gave you above. You might need to use some of them to simplify. One you will see come up often is $\sin^2\theta + \cos^2\theta = 1$.

EXAMPLE: Establish the identity: $\csc\theta \cdot \tan\theta = \sec\theta$.

You want to show that one side of the equation equals the other side. In these problems you are NOT allowed to do operations like adding or subtracting things from one side to the other. Think of each side as independent. We are not going to do anything with the right hand side. On the left side we will use the first technique and change the cosecant and tangent functions into sines and cosines:

$\dfrac{1}{\sin\theta} \cdot \dfrac{\sin\theta}{\cos\theta} = \sec\theta$ We can now cancel the sines from the left side of the equation.

$\dfrac{1}{\cos\theta} = \sec\theta$ We can change the fraction on the left side into secant.

$\sec\theta = \sec\theta$ Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\dfrac{\sin^4\theta - \cos^4\theta}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$.

Another technique I mentioned is factoring. We can factor the top because of difference of squares. Remember that $a^2 - b^2$ factors into $(a+b)(a-b)$. Therefore,

$\sin^4\theta - \cos^4\theta = \left(\sin^2\theta\right)^2 - \left(\cos^2\theta\right)^2 = \left(\sin^2\theta + \cos^2\theta\right)\left(\sin^2\theta - \cos^2\theta\right)$ using difference of squares.

$\dfrac{\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^2\theta - \cos^2\theta\right)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We know $\sin^2\theta + \cos^2\theta = 1$

$\dfrac{\left(\sin^2\theta - \cos^2\theta\right)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We can factor the top again by difference of squares.

$\dfrac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We want to factor a negative out of the first term.

$\dfrac{-(-\sin\theta + \cos\theta)(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ Now switch the order in the first term on top.

$\dfrac{-(\cos\theta - \sin\theta)(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ Now we can cancel the $\cos\theta - \sin\theta$ terms.

$-(\sin\theta + \cos\theta) = -(\cos\theta + \sin\theta)$ Both sides are equal so the proof is done.

EXAMPLE: Establish the identity: $\dfrac{\cos x - 2\sin x\cos x}{\cos^2 x - \sin^2 x + \sin x - 1} = \cot x$.

$\dfrac{\cos x(1 - 2\sin x)}{\cos^2 x - \sin^2 x + \sin x - 1} = \cot x$ First we can factor the numerator. Now we want to get all sines on the bottom. We can use the identity $\cos^2 x = 1 - \sin^2 x$.

$\dfrac{\cos x(1 - 2\sin x)}{(1 - \sin^2 x) - \sin^2 x + \sin x - 1} = \cot x$ Now simplify the denominator.

$\dfrac{\cos x(1 - 2\sin x)}{-2\sin^2 x + \sin x} = \cot x$ Factor the denominator.

$\dfrac{\cos x(1 - 2\sin x)}{\sin x(-2\sin x + 1)} = \cot x$ The part in parenthesis on top and bottom can be cancelled.

$\dfrac{\cos x}{\sin x} = \cot x$ We will use the identity $\cot x = \dfrac{\cos x}{\sin x}$

$\cot x = \cot x$ Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\cot\theta + \dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta$.

Since this problem has a fraction, I will follow technique #3, which says to get common denominators if there are fractions. At the same time I will also use the identity: $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

$\dfrac{\cos\theta}{\sin\theta}\cdot\left(\dfrac{\cos\theta}{\cos\theta}\right) + \dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta$ Now write as a single fraction.

$\dfrac{\cos^2\theta + 1 - 2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta$ Now simplify the numerator.

$\dfrac{1-\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta$ We will now use the identity $\sin^2\theta = 1-\cos^2\theta$.

$\dfrac{\sin^2\theta}{\sin\theta\cos\theta} = \tan\theta$ We can cancel a sine from the top and bottom.

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ We will use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$\tan\theta = \tan\theta$ Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = 2\sec x$.

Once again we want to first get a single fraction so we need common denominators.

$\dfrac{\cos x}{1+\sin x}\cdot\left(\dfrac{\cos x}{\cos x}\right) + \dfrac{1+\sin x}{\cos x}\cdot\left(\dfrac{1+\sin x}{1+\sin x}\right) = 2\sec x$ Now multiply and write as a single fraction.

$\dfrac{\cos^2 x + (1+\sin x)^2}{\cos x(1+\sin x)} = 2\sec x$ We will expand the numerator.

$\dfrac{\cos^2 x + \sin^2 x + 2\sin x + 1}{\cos x(1+\sin x)} = 2\sec x$ We will use the identity $\cos^2 x + \sin^2 x = 1$

$\dfrac{1 + 2\sin x + 1}{\cos x(1+\sin x)} = 2\sec x$ Simplify the numerator.

$\dfrac{2\sin x + 2}{\cos x(1+\sin x)} = 2\sec x$ Factor the numerator.

$\dfrac{2(\sin x+1)}{\cos x(1+\sin x)}=2\sec x$ We can cancel the $\sin x+1$ from the top and bottom.

$\dfrac{2}{\cos x}=2\sec x$ We will use the identity $\sec x=\dfrac{1}{\cos x}$.

$2\sec x=2\sec x$ Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\dfrac{\tan x+\cot x}{\sec x\csc x}=1$.

We will use technique #1 and change everything into sines and cosines. This makes it easier to reduce.

$$\frac{\dfrac{\sin x}{\cos x}+\dfrac{\cos x}{\sin x}}{\dfrac{1}{\cos x}\cdot\dfrac{1}{\sin x}}=1$$ We need to get common denominators in the numerator.

$$\frac{\dfrac{\sin x}{\cos x}\cdot\left(\dfrac{\sin x}{\sin x}\right)+\dfrac{\cos x}{\sin x}\cdot\left(\dfrac{\cos x}{\cos x}\right)}{\dfrac{1}{\cos x}\cdot\dfrac{1}{\sin x}}=1$$ Multiply and write as one fraction in the numerator.

$$\frac{\dfrac{\sin^2 x+\cos^2 x}{\sin x\cos x}}{\dfrac{1}{\sin x\cos x}}=1$$ Now use the identity $\sin^2\theta+\cos^2\theta=1$.

$$\frac{\dfrac{1}{\sin x\cos x}}{\dfrac{1}{\sin x\cos x}}=1$$ Flip over the bottom fraction and multiply.

$$\frac{1}{\sin x\cos x}\cdot\frac{\sin x\cos x}{1}=1$$ We can cancel terms.

$1=1$ Both sides are the same so we are done.

EXAMPLE: Establish the identity: $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}$.

We see that everything is already in terms of sine and cosine. Also notice that we can't factor and even though there are fractions, we don't need common denominators. The only other technique that we can use is technique #4, which says to multiply both sides by a conjugate. We can do this on either side. Please note that we are NOT allowed to cross multiply because we need to treat both sides separately.

$$\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}\cdot\left(\frac{1+\sin\theta}{1+\sin\theta}\right)$$

I chose the right side, but we can chose either side to work with.

$$\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

Now use the identity $\cos^2 x = 1-\sin^2 x$.

$$\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$

Now we can cancel the cosine from the top and bottom.

$$\frac{1+\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

Both sides are equal so we are done.

Let's do this problem again, but now let's work on the left side instead of the right side.

$$\left(\frac{1-\sin\theta}{1-\sin\theta}\right)\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}$$

I chose the left side this time.

$$\frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos\theta}{1-\sin\theta}$$

Now use the identity $\cos^2 x = 1-\sin^2 x$.

$$\frac{\cos^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos\theta}{1-\sin\theta}$$

Now we can cancel the cosine from the top and bottom.

$$\frac{\cos}{1-\sin\theta} = \frac{\cos\theta}{1-\sin\theta}$$

Both sides are equal so we are done.

Remember you only need to show that both sides are equal. What each side is does not matter, as long as both sides are the same. I am not looking for one exact way of doing these problems, because there may be more than one way to show that one side equals the other. Just make sure you logically show your steps. If you start with one statement and then jump down to the answer without showing how you got there, you will not receive full credit. Your answers to these problems will be these logical steps you show.