7.6 Double-Angle and Half-Angle Formulas

If we have either a double angle 2θ or a half angle $\frac{\theta}{2}$ then these have special formulas:

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

There are three formulas for $\cos(2\theta)$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

The above double angle formulas can be manipulated to derive power reducing formulas. These formulas will be useful primarily in Calculus 2:

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Besides double angle formulas there are also half angle formulas:

Half Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

You will choose plus or minus depending on what quadrant $\frac{\theta}{2}$ is.

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

You will choose plus or minus depending on what quadrant $\frac{\theta}{2}$ is.

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

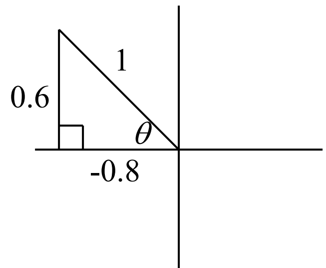
You will choose plus or minus depending on what quadrant $\frac{\theta}{2}$ is.

There are better formulas for $\tan\frac{\theta}{2}$ that don't involve a plus or minus:

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}, \qquad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

EXAMPLE: Compute $\sin\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ if you are given $\cos\theta = -0.8$ and $90^\circ \le \theta \le 180^\circ$. Round decimals to two decimal places.

We need to draw a triangle for this one. We are given that the triangle should be drawn in the second quadrant. We can rewrite our problem as $\cos\theta = \frac{-0.8}{1}$. We know the adjacent side is –0.8. The hypotenuse is 1. Once we label our triangle we find the third side by using the Pythagorean theorem.



Now we can get our first 5 trigonometric functions by reading off our triangle:

$\sin\theta = 0.6$ $\quad\csc\theta = \frac{1}{0.6} = 1.67$ $\quad\sec\theta = \frac{1}{-0.8} = -1.25$

$\tan\theta = \frac{0.6}{-0.8} = -0.75$ $\quad\cot\theta = \frac{-0.8}{0.6} = -1.33$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2\sin\theta\cos\theta$. We already know sine and cosine, so we will substitute in those decimals: $\sin(2\theta) = 2(0.6)(-0.8) = -0.96$.

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2\cos^2\theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos\theta = -0.8$, so we will substitute in this decimal: $\cos(2\theta) = 2(-0.8)^2 - 1 = 0.28$.

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$. We already know $\tan\theta = -0.75$, so we will substitute in this decimal: $\tan(2\theta) = \frac{2(-0.75)}{1-(-0.75)^2} = \frac{-1.5}{0.4375} = -3.43$.

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let's first start with our given statement $90^\circ \le \theta \le 180^\circ$. If I divide everything by two we get: $45^\circ \le \frac{\theta}{2} \le 90^\circ$. This tells us that $\frac{\theta}{2}$ is in the first quadrant, so sine, cosine, and tangent of $\frac{\theta}{2}$ should all be positive.

Now I will find $\sin\frac{\theta}{2}$ by using its formula: $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the first quadrant. We already know $\cos\theta=-0.8$, so we will substitute in this decimal:

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-(-0.8)}{2}}=\sqrt{\frac{1.8}{2}}=\sqrt{.9}=0.95.$$

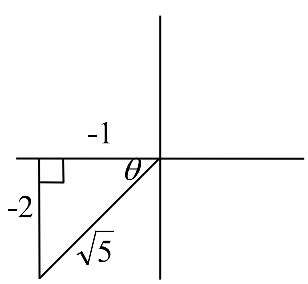
Now I will find $\cos\frac{\theta}{2}$ by using its formula: $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the first quadrant. We already know $\cos\theta=-0.8$, so we will substitute in this decimal:

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+(-0.8)}{2}}=\sqrt{\frac{0.2}{2}}=\sqrt{.1}=0.32.$$

Finally I will find $\tan\frac{\theta}{2}$. I have three formulas to choose. I will choose: $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$. We already know $\cos\theta=-0.8$, and $\sin\theta=0.6$ so we will substitute in these decimals: $\tan\frac{\theta}{2}=\frac{0.6}{1+(-0.8)}=3$.

EXAMPLE: Compute $\sin\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ if you are given $\cot\theta=\frac{1}{2}$ and $\sin\theta<0$.

We are given than the cotangent is positive and sine is negative. This only occurs in the 3rd quadrant. We know the adjacent side is 1 and the opposite side is 2. However, because we are in the third quadrant we need to make the 1 and 2 negative. Once we label our triangle we find the third side by using the Pythagorean theorem. This will give us $\sqrt{5}$.



Now we can get our first 5 trigonometric functions by reading off our triangle:

$\sin\theta=-\frac{2}{\sqrt{5}}=-\frac{2\sqrt{5}}{5}$ $\cos\theta=-\frac{1}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$ $\sec\theta=-\sqrt{5}$

$\csc\theta=-\frac{\sqrt{5}}{2}$ $\tan\theta=2$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2\sin\theta\cos\theta$. We already know sine and cosine, so we will substitute in those fractions: $\sin(2\theta) = 2\left(-\frac{2\sqrt{5}}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) = \frac{20}{25} = \frac{4}{5}$.

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2\cos^2\theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos\theta = -\frac{\sqrt{5}}{5}$, so we will substitute in this fraction: $\cos(2\theta) = 2\left(-\frac{\sqrt{5}}{5}\right)^2 - 1$. This simplifies to: $2\left(\frac{5}{25}\right) - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$.

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$. We already know $\tan\theta = 2$, so we will substitute in this number: $\tan(2\theta) = \frac{2(2)}{1-(2)^2} = -\frac{4}{3}$.

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let's first start with our given statement $180^\circ \le \theta \le 270^\circ$. If I divide everything by two we get: $90^\circ \le \frac{\theta}{2} \le 135^\circ$. This tells us that $\frac{\theta}{2}$ is in the second quadrant, so sine of $\frac{\theta}{2}$ should be positive, and the cosine and tangent of $\frac{\theta}{2}$ should be negative.

Now I will find $\sin\frac{\theta}{2}$ by using its formula: $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos\theta = -0.8$, so we will substitute in this decimal:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\left(-\frac{\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}} = \sqrt{\frac{5+\sqrt{5}}{10}}.$$

Now I will find $\cos\frac{\theta}{2}$ by using its formula: $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$. We chose a negative because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos\theta = -0.8$, so we will substitute in this decimal:

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\left(-\frac{\sqrt{5}}{5}\right)}{2}} = -\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}} = -\sqrt{\frac{5-\sqrt{5}}{10}}.$$

Finally I will find $\tan\frac{\theta}{2}$. I have three formulas to choose. I will choose: $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$. We already know $\cos\theta=-0.8$, and $\sin\theta=0.6$ so we will substitute in these decimals:

$$\tan\frac{\theta}{2}=\frac{-\frac{2\sqrt{5}}{5}}{1+\left(-\frac{\sqrt{5}}{5}\right)}=\frac{-\frac{2\sqrt{5}}{5}}{\frac{5-\sqrt{5}}{5}}=-\frac{2\sqrt{5}}{5}\cdot\frac{5}{5-\sqrt{5}}=-\frac{2\sqrt{5}}{5-\sqrt{5}}\cdot\frac{5+\sqrt{5}}{5+\sqrt{5}}=\frac{-10\sqrt{5}-10}{25-5}=\frac{-10\sqrt{5}-10}{20}=\frac{-\sqrt{5}-1}{2}.$$

EXAMPLE: Compute $\sin(22.5^\circ)$ and $\tan(22.5^\circ)$ using a half-angle formula.

We can write this as $\sin\left(\frac{45^\circ}{2}\right)$. Then we know that θ is 45 degrees, so now we use: $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos 45^\circ}{2}}$. It is positive since 22.5 is in the first quadrant. You get: $\sin\frac{\theta}{2}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{\sqrt{2-\sqrt{2}}}{2}$.

For $\tan(22.5^\circ)$ we will use: $\tan\frac{45^\circ}{2}=\frac{\sin 45^\circ}{1+\cos 45^\circ}=\frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}=\frac{\frac{\sqrt{2}}{2}}{\frac{2+\sqrt{2}}{2}}=\frac{\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}=\frac{2\sqrt{2}-2}{2}=\sqrt{2}-1$.

EXAMPLE: Rewrite $1-2\sin^2(\pi/12)$ using a double angle formula. Then find its exact value.

This one is written in the form $1-2\sin^2\theta$, which is a form of $\cos 2\theta$. In this problem, we will let $\theta=\pi/12$. Therefore we will substitute this into the formula $\cos 2\theta$: $\cos\left[2\left(\frac{\pi}{12}\right)\right]$. Simplifying gives us $\cos\frac{\pi}{6}$. Looking at our table we see this has an exact value of $\sqrt{3}/2$.

EXAMPLE: Establish the identity: $\frac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta}=\cos 2\theta$

First we want to change these into sines and cosines.

$$\frac{\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}}=\cos 2\theta$$

Now get common denominators on the top and bottom.

$$\frac{\left(\frac{\cos\theta}{\cos\theta}\right)\cdot\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\cdot\left(\frac{\sin\theta}{\sin\theta}\right)}{\left(\frac{\cos\theta}{\cos\theta}\right)\cdot\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\cdot\left(\frac{\sin\theta}{\sin\theta}\right)}=\cos 2\theta$$

Multiply and write over a single denominator

$$\frac{\frac{\cos^2\theta - \sin^2\theta}{\cos\theta\sin\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos\theta\sin\theta}} = \cos 2\theta$$

We will get rid of the double fractions and use $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$ The left side is the identity for $\cos 2\theta$.

$\cos 2\theta = \cos 2\theta$

EXAMPLE: $(4\sin\theta\cos\theta)(1 - 2\sin^2\theta) = \sin 4\theta$

First we will use the identity $1 - 2\sin^2\theta = \cos 2\theta$. Now our problem becomes:

$(4\sin\theta\cos\theta)\cos 2\theta = \sin 4\theta$

We know that $\sin 2\theta = 2\sin\theta\cos\theta$. We want to rewrite our problem as the following:

$2 \cdot 2\sin\theta\cos\theta \cdot \cos 2\theta = \sin 4\theta$ Now we can use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

$2\sin 2\theta\cos 2\theta = \sin 4\theta$ We know that $2\sin 2\theta\cos 2\theta$ is the same as $\sin(2 \cdot 2\theta) = \sin 4\theta$.

$\sin 4\theta = \sin 4\theta$

EXAMPLE: Use power-reducing formulas to rewrite $\cos^4\theta$ in terms of first powers of cosine.

First we can rewrite the original problem as: $\cos^2\theta \cdot \cos^2\theta$. This will allow us to use the power reducing formula for cosine. We will use the same formula twice: $\frac{1+\cos(2\theta)}{2} \cdot \frac{1+\cos(2\theta)}{2}$. Now multiply across the top and bottom to get $\frac{1 + 2\cos(2\theta) + \cos^2(2\theta)}{4}$. We can split up the fraction: $\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)$. Now we will use another power reducing formula: $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$. When you substitute this in you will get $\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4} \cdot \frac{1+\cos(4\theta)}{2}$. Multiply to get $\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1+\cos(4\theta)}{8}$. Now split up the last fraction: $\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{8} + \frac{\cos(4\theta)}{8}$. Add the fractions to get the final answer: $\frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$