

7.7 Product-to-Sum and Sum-to-Product Formulas

Product-to-Sum Formulas

$$\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)]$$

$$\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)]$$

$$\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)]$$

$$\cos x \sin y = \frac{1}{2} [\sin(x + y) - \sin(x - y)]$$

EXAMPLE: Simplify: $\sin(6\theta)\sin(4\theta)$ using a product-to-sum formula.

We will use the first formula $\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)]$ with $x = 6\theta$ and $y = 4\theta$

$$\frac{1}{2} [\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)] \quad \text{Now simplify.}$$

$$\frac{1}{2} [\cos(2\theta) - \cos(10\theta)] \quad \text{This is as far as we can go.}$$

EXAMPLE: Simplify: $\cos(3\theta)\cos(\theta)$ using a product-to-sum formula.

We will use the formula $\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)]$ with $x = 3\theta$ and $y = \theta$.

$$\frac{1}{2} [\cos(3\theta - \theta) + \cos(3\theta + \theta)] \quad \text{Now simplify.}$$

$$\frac{1}{2} [\cos(2\theta) + \cos(4\theta)] \quad \text{This is as far as we can go.}$$

EXAMPLE: Simplify: $\sin(3\theta)\cos(5\theta)$ using a product-to-sum formula.

We will use the formula $\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)]$ with $x = 3\theta$ and $y = 5\theta$.

$$\frac{1}{2} [\sin(3\theta - 5\theta) + \sin(3\theta + 5\theta)] \quad \text{Now simplify.}$$

$$\frac{1}{2} [\sin(-2\theta) + \sin(8\theta)] \quad \text{We will use the identity } \sin(-2\theta) = -\sin 2\theta.$$

$$\frac{1}{2} [-\sin(2\theta) + \sin(8\theta)] \quad \text{or} \quad \frac{1}{2} [\sin(8\theta) - \sin(2\theta)] \quad \text{is as far as we can go.}$$

EXAMPLE: Find the exact value of $\cos \frac{5\pi}{12} \sin \frac{\pi}{12}$ using a product-to-sum formula.

We will use the formula $\cos x \sin y = \frac{1}{2} [\sin(x+y) - \sin(x-y)]$ with $x = \frac{5\pi}{12}$ and $y = \frac{\pi}{12}$.

$$\frac{1}{2} \left[\sin \left(\frac{5\pi}{12} + \frac{\pi}{12} \right) - \sin \left(\frac{5\pi}{12} - \frac{\pi}{12} \right) \right] \quad \text{Now simplify.}$$

$$\frac{1}{2} \left[\sin \left(\frac{\pi}{2} \right) - \sin \left(\frac{\pi}{3} \right) \right] \quad \text{We can use our table to get the values of these trig functions.}$$

$$\frac{1}{2} \left[1 - \frac{\sqrt{3}}{2} \right] = \frac{1}{2} - \frac{\sqrt{3}}{4} = \frac{2 - \sqrt{3}}{4}.$$

Sum-to-Product Formulas

$$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\sin x - \sin y = 2 \sin \left(\frac{x-y}{2} \right) \cos \left(\frac{x+y}{2} \right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

EXAMPLE: Simplify: $\sin(5\theta) - \sin(3\theta)$ using a sum-to-product formula.

We will use the formula $\sin x - \sin y = 2 \sin \left(\frac{x-y}{2} \right) \cos \left(\frac{x+y}{2} \right)$ with $x = 5\theta$ and $y = 3\theta$.

$$2 \sin \left(\frac{5\theta - 3\theta}{2} \right) \cos \left(\frac{5\theta + 3\theta}{2} \right) \quad \text{Now simplify.}$$

$$2 \sin \left(\frac{2\theta}{2} \right) \cos \left(\frac{8\theta}{2} \right)$$

$$2 \sin(\theta) \cos(4\theta) \quad \text{We can't simplify this anymore, so we are done.}$$

EXAMPLE: Simplify: $\cos(3\theta) + \cos(2\theta)$ using a sum-to-product formula.

We will use the formula $\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$ with $x = 3\theta$ and $y = 2\theta$.

$$2 \cos\left(\frac{3\theta + 2\theta}{2}\right) \cos\left(\frac{3\theta - 2\theta}{2}\right) \quad \text{Now simplify.}$$

$$2 \cos\left(\frac{5\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \quad \text{We can't simplify anymore, so we are done.}$$

EXAMPLE: Simplify: $\cos(4\theta) - \cos(7\theta)$ using a sum-to-product formula.

We will use the formula $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$ with $x = 4\theta$ and $y = 7\theta$.

$$-2 \sin\left(\frac{4\theta + 7\theta}{2}\right) \sin\left(\frac{4\theta - 7\theta}{2}\right) \quad \text{Now simplify.}$$

$$-2 \sin\left(\frac{11\theta}{2}\right) \sin\left(\frac{-3\theta}{2}\right) \quad \text{We can use the identity } \sin\left(\frac{-3\theta}{2}\right) = -\sin\left(\frac{3\theta}{2}\right).$$

$$-2 \sin\left(\frac{11\theta}{2}\right) \cdot -\sin\left(\frac{3\theta}{2}\right)$$

$$2 \sin\left(\frac{11\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)$$

EXAMPLE: Find the exact value of $\sin 15^\circ + \sin 75^\circ$ using a sum-to-product formula.

We will use the formula $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$ with $x = 15^\circ$ and $y = 75^\circ$.

$$2 \sin\left(\frac{15^\circ + 75^\circ}{2}\right) \cos\left(\frac{15^\circ - 75^\circ}{2}\right) \quad \text{Simplify.}$$

$$2 \sin(45^\circ) \cos(-30^\circ) \quad \text{From here we can use our table to get the exact values.}$$

$$2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}.$$