8.3 Partial Fractions

Suppose we were asked to write the following as a single fraction: $\frac{4}{x+1}-\frac{3}{x-2}$. We would need to get common denominators: $\frac{4}{x+1}\cdot\left(\frac{x-2}{x-2}\right)-\frac{3}{x-2}\cdot\left(\frac{x+1}{x+1}\right)$. You will get: $\frac{4(x-1)-3(x+1)}{(x+1)(x-2)}$. Distributing on top will give you: $\frac{4x-8-3x+3}{(x+1)(x-2)}$. This simplifies to: $\frac{x-11}{(x+1)(x-2)}$. Therefore we come to the following: $\frac{x-11}{(x+1)(x-2)}=\frac{4}{x+1}-\frac{3}{x-2}$. In this section we will be doing the reverse of what we just did. We will start with the single fraction and break it up (decompose it) into separate fractions. This is something you will do in a calculus 2 course. Sometimes it is easier to do calculus operations on two smaller fractions instead of one big fraction.

Three Rules of How a Fraction Decomposes

Let $P(x)$ be a polynomial.

Rule 1: $\frac{P(x)}{(x-a_1)(x-a_2)...(x-a_n)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+...+\frac{A_n}{x-a_n}$

Rule 2: $\frac{P(x)}{(x-a)^n}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+...+\frac{A_n}{(x-a)^n}$

Rule 3: $\frac{P(x)}{\left(ax^2+bx+c\right)^n}=\frac{A_1x+B_1}{\left(ax^2+bx+c\right)}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+...+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$

Where ax^2+bx+c is irreducible, or nonfactorable.

EXAMPLE: Set up the following for decomposition but DO NOT SOLVE: $\frac{3x-5}{x^2-6x+5}$.

We first need to factor: $\frac{3x-5}{(x-5)(x-1)}$. Now we will use rule 1: $\frac{3x-5}{(x-5)(x-1)}=\frac{A_1}{x-5}+\frac{A_2}{x-1}$.

EXAMPLE: Set up the following for decomposition but DO NOT SOLVE: $\frac{2x-4}{\left(x^2+x+1\right)^3}$.

This time we can't factor the denominator. This means we need to use rule 3:

$$\frac{2x-4}{\left(x^2+x+1\right)^3}=\frac{A_1x+B_1}{\left(x^2+x+1\right)}+\frac{A_2x+B_2}{\left(x^2+x+1\right)^2}+\frac{A_3x+B_3}{\left(x^2+x+1\right)^3}$$

EXAMPLE: Set up the following for decomposition but DO NOT SOLVE: $\dfrac{2x^2-1}{(3x-5)^4}$.

This one requires rule 2: $\dfrac{2x^2-1}{(3x-5)^4}=\dfrac{A_1}{3x-5}+\dfrac{A_2}{(3x-5)^2}+\dfrac{A_3}{(3x-5)^3}+\dfrac{A_4}{(3x-5)^4}$

EXAMPLE: Set up the following for decomposition but DO NOT SOLVE: $\dfrac{4x^2-3x+1}{x(x-3)^2}$.

Sometimes you may need to combine more than one rule. This one will use rule 1 and rule 2:
$\dfrac{4x^2-3x+1}{x(x-3)^2}=\dfrac{A_1}{x}+\dfrac{A_2}{x-3}+\dfrac{A_3}{(x-3)^2}$

EXAMPLE: Determine the partial fraction decomposition: $\dfrac{5x+27}{x^2-9}$.

The first thing we should do with all these kind of problems is to factor if possible: $\dfrac{5x+27}{(x+3)(x-3)}$. Now we can use rule 1 to set it up: $\dfrac{5x+27}{(x+3)(x-3)}=\dfrac{A_1}{x+3}+\dfrac{A_2}{x-3}$. Now we want to get common denominators with the right side of the equation: $\dfrac{5x+27}{(x+3)(x-3)}=\dfrac{A_1}{x+3}\cdot\left(\dfrac{x-3}{x-3}\right)+\dfrac{A_2}{x-3}\cdot\left(\dfrac{x+3}{x+3}\right)$. This will give us:

$\dfrac{5x+27}{(x+3)(x-3)}=\dfrac{A_1(x-3)+A_2(x+3)}{(x+3)(x-3)}$. Since the denominator of each fraction is the same, we can just set the numerators equal to each other. You will get the following equation: $5x+27=A_1(x-3)+A_2(x+3)$. From here the book shows two methods of solving this. I will only focus on one method. This one involves choosing a value for x and plugging it into both sides of the equation. You want to choose a number that will cancel part of the equation so you only have one variable left to solve for. In the problem above I want to let x be 3 and -3.

Let x = 3. We will put 3 into both sides of the equation: $5(3)+27=A_1(3-3)+A_2(3+3)$. You want to simplify inside the parenthesis first: $42=A_1(0)+A_2(6)$. This simplifies further to: $42=6A_2$. Solving for $A_2=7$.

Let x = -3. We will put 3 into both sides of the equation: $5(-3)+27=A_1(-3-3)+A_2(-3+3)$. You want to simplify inside the parenthesis first: $12=A_1(-6)+A_2(0)$. This simplifies further to: $12=-6A_1$. Solving for A_1 we get: $A_1=-2$.

Our answer is written as: $\dfrac{5x+27}{(x+3)(x-3)}=\dfrac{-2}{x+3}+\dfrac{7}{x-3}$.

EXAMPLE: Determine the partial fraction decomposition: $\dfrac{6-4x}{x^3-x^2-4x+4}$.

The first thing we should do is to factor the denominator. This involves the grouping method. Factor the first two terms and the second two terms separately: $x^2(x-1)-4(x-1)$. Now factor out the common factor of x – 1: $(x-1)(x^2-4)$. We can factor this one more time to get: $(x-1)(x-2)(x+2)$. So now our problem becomes: $\dfrac{6-4x}{(x-1)(x+2)(x-2)}$. Now we can use rule 1 to set it up:

$\dfrac{6-4x}{(x-1)(x+2)(x-2)}=\dfrac{A_1}{x-1}+\dfrac{A_2}{x+2}+\dfrac{A_3}{x-2}$. Now we want to get common denominators

$\dfrac{6-4x}{(x-1)(x+2)(x-2)}=\dfrac{A_1}{x-1}\cdot\dfrac{(x+2)(x-2)}{(x+2)(x-2)}+\dfrac{A_2}{x+2}\cdot\dfrac{(x-1)(x-2)}{(x-1)(x-2)}+\dfrac{A_3}{x-2}\cdot\dfrac{(x-1)(x+2)}{(x-1)(x+2)}$. This will give us:

$\dfrac{6-4x}{(x-1)(x+2)(x-2)}=\dfrac{A_1(x+2)(x-2)+A_2(x-1)(x-2)+A_3(x-1)(x+2)}{(x-1)(x+2)(x-2)}$. Since the denominator of each fraction is the same, we can just set the numerators equal to each other. You will get the following equation: $6-4x=A_1(x+2)(x-2)+A_2(x-1)(x-2)+A_3(x-1)(x+2)$. You want to choose a number that will cancel part of the equation so you only have one variable left to solve for. In the problem above I want to let x be 1, 2, and -2.

Let x = 1. We will put 3 in for x: $6-4(1)=A_1(1+2)(1-2)+A_2(1-1)(1-2)+A_3(1-1)(1+2)$. You want to simplify inside the parenthesis first: $2=A_1(3)(-1)+0+0$. This simplifies further to: $2=-3A_2$. Solving for A_1 we get: $A_1=-2/3$.

Let x = 2. We will put 2 in for x: $6-4(2)=A_1(2+2)(2-2)+A_2(2-1)(2-2)+A_3(2-1)(2+2)$. You want to simplify inside the parenthesis first: $-2=0+0+A_3(1)(4)$. This simplifies further to: $-2=4A_2$. Solving for A_2 we get: $A_3=-1/2$.

Let x = -2. We will put 3 in for x: $6-4(-2)=A_1(-2+2)(-2-2)+A_2(-2-1)(-2-2)+A_3(-2-1)(-2+2)$. You want to simplify inside the parenthesis first: $14=0+A_2(-3)(-4)+0$. This simplifies further to: $14=12A_2$. Solving for A_2 we get: $A_2=7/6$.

Our answer is written as: $\dfrac{6-4x}{(x-1)(x+2)(x-2)}=\dfrac{-2/3}{x-1}+\dfrac{7/6}{x+2}+\dfrac{-1/2}{x-2}$ or $\dfrac{-2}{3(x-1)}+\dfrac{7}{6(x+2)}-\dfrac{1}{2(x-2)}$

EXAMPLE: Determine the partial fraction decomposition: $\dfrac{7x}{(x-5)^2}$.

We can use rule 2 to set this up: $\dfrac{7x}{(x-5)^2}=\dfrac{A_1}{x-5}+\dfrac{A_2}{(x-5)^2}$. Now we want to get common denominators with the right side of the equation: $\dfrac{7x}{(x-5)^2} = \dfrac{A_1}{x-5} \cdot \left(\dfrac{x-5}{x-5}\right) + \dfrac{A_2}{(x-5)^2}$. This will give us:

$\dfrac{7x}{(x-5)^2} = \dfrac{A_1(x-5)+A_2}{(x-5)^2}$. Since the denominator of each fraction is the same, we can just set the numerators equal to each other. You will get the following equation: $7x = A_1(x-5)+A_2$. You want to choose a number that will cancel part of the equation so you only have one variable left to solve for. In the problem above I want to let x be 5.

Let x = 5. We will put 5 into both sides of the equation: $7(5) = A_1(5-5)+A_2$. You want to simplify inside inside the parenthesis first: $35 = A_1(0)+A_2$. This simplifies further to: $35 = A_2$.

So now our problem becomes: $7x = A_1(x-5)+35$. We don't have another value to plug in for x to cancel something out. So now we can let x be ANY number. It doesn't matter which one you choose because you will get the same answer for A_1. Let's let x = 0 since this is an easy one to plug in:

Let x = 0. We will put 0 into both sides of the equation: $7(0) = A_1(0-5)+35$. You want to simplify inside the parenthesis first: $0 = -5A_1 + 35$. Solving for A_1 we get: $A_1 = 7$.

Our answer is written as: $\dfrac{7x}{(x-5)^2} = \dfrac{7}{x-5} + \dfrac{35}{(x-5)^2}$.

EXAMPLE: Determine the partial fraction decomposition: $\dfrac{x+2}{x^3-x^2}$.

We want to factor a common factor out of the denominator. We will get: $\dfrac{x+2}{x^2(x-1)}$. Now we can use rule 1 and rule 2 to set it up: $\dfrac{x+2}{x^2(x-1)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{A_3}{x-1}$. Now we want to get common denominators

$\dfrac{x+2}{x^2(x-1)} = \dfrac{A_1}{x} \cdot \dfrac{x(x-1)}{x(x-1)} + \dfrac{A_2}{x^2} \cdot \dfrac{x-1}{x-1} + \dfrac{A_3}{x-1} \cdot \dfrac{x^2}{x^2}$. This will give us: $\dfrac{x+2}{x^2(x-1)} = \dfrac{A_1x(x-1)+A_2(x-1)+A_3x^2}{x^2(x-1)}$.

Since the denominator of each fraction is the same, we can just set the numerators equal to each other. You will get the following equation: $x+2 = A_1 \cdot x(x-1) + A_2(x-1) + A_3x^2$. You want to choose a number that will cancel part of the equation so you only have one variable left to solve for. In the problem above I want to let x be 0, and 1.

Let x = 0. We will put 0 in for x: $0+2 = A_1 \cdot 0(0-1) + A_2(0-1) + A_3 0^2$. You want to simplify inside the parenthesis first: $2 = 0 + A_2(-1) + 0$. This simplifies further to: $2 = -A_2$. Solving for A_2 we get: $A_2 = -2$.

Let x = 1. We will put 1 in for x: $1+2 = A_1 \cdot 1(1-1) + A_2(1-1) + A_3 1^2$. You want to simplify inside the parenthesis first: $3 = 0 + 0 + A_3 \cdot 1$. This gives us: $A_3 = 3$.

So now our problem becomes: $x+2 = A_1 \cdot x(x-1) - 2(x-1) + 3x^2$ Since we have run out of numbers to plug in for x to cancel terms, we can now choose ANY number to plug in. I will choose x = 2.

Let x = 2. We will put 3 in for x: $2+2 = A_1 \cdot 2(2-1) - 2(2-1) + 3 \cdot 2^2$. You want to simplify inside the parenthesis first: $4 = 2A_1 - 2 + 12$. This simplifies further to: $4 = 2A_1 + 10$. Solving for A_1 we get: $A_1 = -3$.

Our answer is written as: $\dfrac{x+2}{x^2(x-1)} = \dfrac{-3}{x} - \dfrac{2}{x^2} + \dfrac{3}{x-1}$.

EXAMPLE: Determine the partial fraction decomposition: $\dfrac{x^2+3}{x^3+6x^2+9x}$.

We want to factor a common factor out of the denominator. We will get: $\dfrac{x^2+3}{x(x^2+6x+9)}$. We can factor this one more time: $\dfrac{x^2+3}{x(x+3)^2}$ Now we can use rule 1 and rule 2 to set it up: $\dfrac{x^2+3}{x(x+3)^2} = \dfrac{A_1}{x} + \dfrac{A_2}{x+3} + \dfrac{A_3}{(x+3)^2}$.

Now we want to get common denominators:

$\dfrac{x^2+3}{x(x+3)^2} = \dfrac{A_1}{x} \cdot \dfrac{(x+3)^2}{(x+3)^2} + \dfrac{A_2}{x+3} \cdot \dfrac{x(x+3)}{x(x+3)} + \dfrac{A_3}{(x+3)^2} \cdot \dfrac{x}{x}$. This will give us:

$$\frac{x^2+3}{x(x+3)^2} = \frac{A_1(x+3)^2 + A_2 x(x+3) + A_3 x}{x(x+3)^2}.$$

Setting the numerators equal we get: $x^2 + 3 = A_1 \cdot (x+3)^2 + A_2 x(x+3) + A_3 x$. Now choose values for x.

Let x = 0. We will put 0 in for x: $0^2 + 3 = A_1 \cdot (0+3)^2 + A_2 \cdot 0(0+3) + A_3 \cdot 0$. This gives us $3 = 9A_1$, so $A_1 = 1/3$.

Let x = -3. We will put -3 in for x: $(-3)^2 + 3 = A_1 \cdot (-3+3)^2 + A_2(-3)(-3+3) + A_3(-3)$. This gives us: $12 = -3A_3$, so $A_3 = -4$

So now our problem becomes: $x^2 + 3 = (1/3) \cdot (x+3)^2 + A_2 x(x+3) - 4x$ Since we have run out of numbers to plug in for x to cancel terms, we can now choose ANY number to plug in. I will choose x = 3.

Let x = 3. We will put 3 in for x: $3^2 + 3 = (1/3) \cdot (3+3)^2 + A_2 \cdot 3(3+3) - 4(3)$. This gives us $12 = 12 + 18A_2 - 12$. Solving for A_2 we get $A_2 = 2/3$

Our answer is: $\dfrac{x^2+3}{x(x+3)^2} = \dfrac{1/3}{x} + \dfrac{2/3}{x+3} - \dfrac{4}{(x+3)^2}$, or you can write: $\dfrac{x^2+3}{x(x+3)^2} = \dfrac{1}{3x} + \dfrac{2}{3(x+3)} - \dfrac{4}{(x+3)^2}$.

EXAMPLE: Determine the partial fraction decomposition: $\frac{x-7}{x^3+2x}$.

We want to factor a common factor out of the denominator. We will get: $\frac{x-7}{x(x^2+2)}$. Now we can use rule 1 and rule 3 to set it up: $\frac{x-7}{x(x^2+2)} = \frac{A_1}{x} + \frac{A_2 x + B_2}{x^2+2}$. Now we want to get common denominators

$\frac{x-7}{x(x^2+2)} = \frac{A_1}{x} \cdot \frac{x^2+2}{x^2+2} + \frac{A_2 x + B_2}{x^2+2} \cdot \frac{x}{x}$. This will give us:

$\frac{x-7}{x(x^2+2)} = \frac{A_1(x^2+2) + (A_2 x + B_2)x}{x(x^2+2)}$.

Setting the numerators equal we get: $x - 7 = A_1 \cdot (x^2+2) + (A_2 x + B_2)x$. Now choose values for x.

Let x = 0. We will put 0 in for x: $0 - 7 = A_1 \cdot (0^2+2) + (A_2(0) + B_2)(0)$. This gives us $-7 = 2A_1$, so $A_1 = -\frac{7}{2}$.

So now our problem becomes: $x - 7 = \frac{-7(x^2+2)}{2} + (A_2 x + B_2)x$ Since we have run out of numbers to plug in for x to cancel terms, we can now choose ANY number to plug in. I will choose x = 2.

Let x = 2. We will put 3 in for x: $2 - 7 = \frac{-7(2^2+2)}{2} + (A_2(2) + B_2)(2)$. This gives us $-5 = -21 + 4A_2 + 2B_2$. We can write this as: $4A_2 + 2B_2 = 16$. This equation can be reduced to: $2A_2 + B_2 = 8$.

But we don't have enough information to solve for the two unknowns, so we need one more equation. We need to pick another value for x. I will let x = 4.

Let x = 4. We will put 3 in for x: $4 - 7 = \frac{-7(4^2+2)}{2} + (A_2(4) + B_2)(4)$. This gives us $-3 = -63 + 16A_2 + 4B_2$. We can write this as: $16A_2 + 4B_2 = 60$. This equation can be reduced to $4A_2 + B_2 = 15$.

We now need to solve the system: $\begin{matrix} 2A_2 + B_2 = 8 \\ 4A_2 + B_2 = 15 \end{matrix}$. If we subtract the two equations we get $-2A_2 = -7$, so $A_2 = 7/2$. Going to the top equation we have $2(7/2) + B_2 = 8$, or $7 + B_2 = 8$. Then $B_2 = 1$.

Our answer is: $\frac{x-7}{x(x^2+2)} = \frac{-7/2}{x} + \frac{(7/2)x+1}{x^2+2}$, or you can write: $\frac{x-7}{x(x^2+2)} = \frac{-7}{2x} + \frac{7x+2}{2(x^2+2)}$.