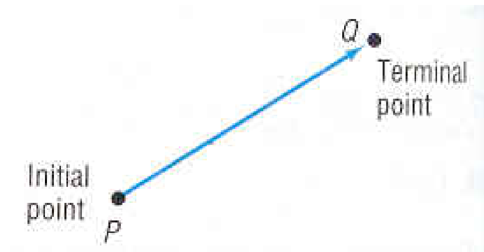
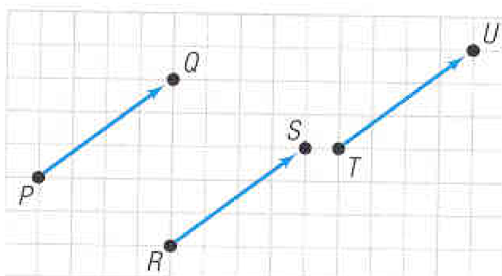
9.4 Vectors

Vectors are needed in physics and engineering courses. A vector is a quantity that has magnitude (size) in a certain direction. You indicate a vector by a ray. The length of the arrow represents the magnitude and the arrowhead indicates the direction of the vector as shown below:

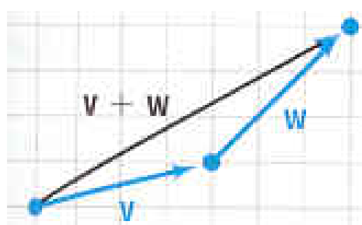


In the picture above the vector starts at point P and ends at Q. It is possible for two vectors to be the same if their magnitude and direction are the same, as shown below.

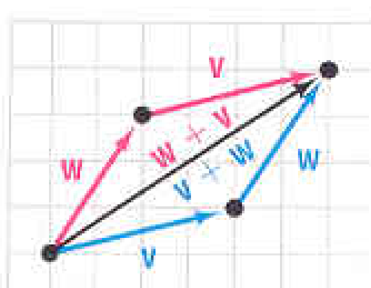


Notice that vector RS and TU are the same as vector PQ.

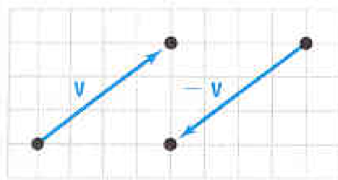
Adding Vectors – when you add two vectors, add them tip to tail as show below:



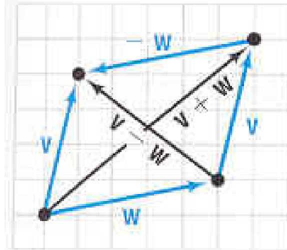
There is more than one way to go from one point to another as shown below. Notice we are still adding the vectors tip to tail:



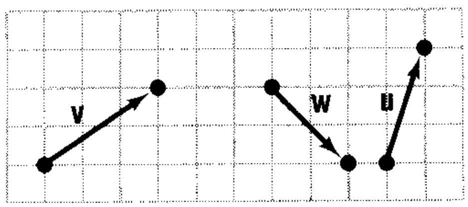
It is possible to have a negative vector. It is drawn the same as the original vector except we change the arrowhead so it goes in the opposite direction:



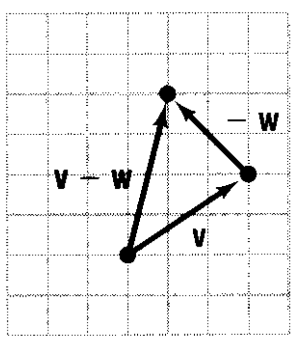
In order to subtract vectors, it is the same thing as adding the opposite: $v - w = v + (-w)$



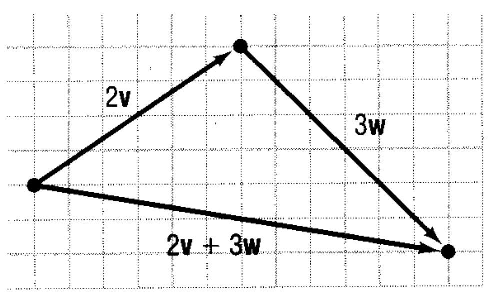
EXAMPLE: Use the following given vectors and sketch the following: a.) $\boldsymbol{v} - \boldsymbol{w}$ b.) $2\boldsymbol{v} + 3\boldsymbol{w}$ c.) $2\boldsymbol{v} - \boldsymbol{w} + \boldsymbol{u}$



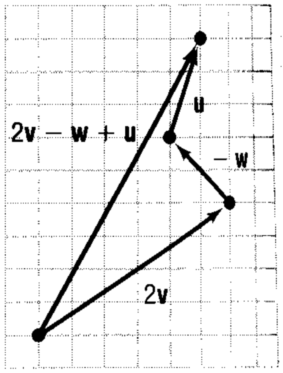
a.) This problem is the same as taking v and adding $-\boldsymbol{w}$. First draw v the same way it originally was drawn. We are moving to the right three boxes and up two boxes. Then from the tip of v you want to draw $-\boldsymbol{w}$. Then we draw a vector from the starting point to the ending point which will be $\boldsymbol{v} - \boldsymbol{w}$.



b.) Since we have $2\boldsymbol{v}$ this means we double the length of the original v. We will first start with this one. Next from the tip of $2\boldsymbol{v}$ we will draw $3\boldsymbol{w}$, which is really 3 of the w vectors. Finally we will draw a vector from our starting point to the ending point, and this will be $2\boldsymbol{v} + 3\boldsymbol{w}$.



c.) There are a couple of different ways to draw this one. I will first start with 2v. Then I will place –w at the tip of 2v. Then on the tip of –w I will place u. Then I will draw a vector from the starting point to the ending point, and this will be 2***v*** – ***w*** + ***u.***



Algebraic Vector

This is expressed as $v = \langle a,\ b \rangle$. The a is the horizontal component and b is the vertical component.

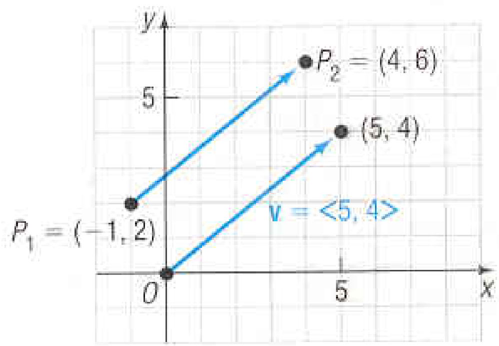
Position Vector

This is an algebraic vector with the starting point at the origin. Suppose that v is a vector with a starting point of $P_1 = (x_1,\ y_1)$ and ending point of $P_2 = (x_2,\ y_2)$. Then $v = \langle x_2 - x_1,\ y_2 - y_1 \rangle$.

EXAMPLE: Find the position vector of the vector $v = P_1P_2$ if $P_1 = (-1,\ 2)$ and $P_2 = (4,\ 6)$. Graph your answer in standard position (graph your position vector).

We will use the formula $v = \langle x_2 - x_1,\ y_2 - y_1 \rangle$ and plug in the given information: $v = \langle 4 - (-1),\ 6 - 2 \rangle$. Then we will simplify: $v = \langle 5,\ 4 \rangle$.

Now we will graph. As in the picture we started with a vector not at the origin and we found an equivalent vector that starts at the origin, and this is the one they want us to graph. Notice these vectors are the same length and are going in the same direction except they start at a different place.



EXAMPLE: Find the position vector of the vector $v = P_1P_2$ if $P_1 = (-1,\ 4)$ and $P_2 = (6,\ -2)$.

We will use the formula $v = \langle x_2 - x_1,\ y_2 - y_1 \rangle$ and plug in the given information: $v = \langle 6 - (-1),\ -2 - 4 \rangle$. Then we will simplify: $v = \langle 7,\ -6 \rangle$.

EXAMPLE: Given $r = \langle 2,\ 5\rangle$, $s = \langle -2,\ -3\rangle$, $w = \langle -2,\ 5\rangle$, find 3**r** – (**w** + **s**).

Let's first substitute in the vectors: $3\langle 2,\ 5\rangle - (\langle -2,\ 5\rangle + \langle -2,\ -3\rangle)$. Now we will distribute and then add the vectors together inside the parenthesis. To add or subtract, add the first components and second components separately: $\langle 6,\ 15\rangle - (\langle -4,\ 2\rangle)$. Finally we subtract the remaining vectors: $\langle 6-(-4),\ 15-2\rangle = \langle 10,\ 13\rangle$.

Alternate form of writing $v = \langle a,\ b\rangle$

We can also write this as v = a**i** + b**j**

How to find a vector's magnitude $\|v\|$

To find the magnitude, use the formula $\|v\| = \sqrt{a^2 + b^2}$

EXAMPLE: Given v = 6**i** + 3**j**, find $\|v\|$.

Here a is 6 and b is 3. We will use the formula $\|v\| = \sqrt{a^2 + b^2}$ and put in our given information: $\|v\| = \sqrt{(6)^2 + (3)^2}$. Now square the terms: $\|v\| = \sqrt{36+9}$. We will get $\|v\| = \sqrt{45}$, so $\|v\| = 3\sqrt{5}$.

EXAMPLE: Given v = **i** + 2**j** and w = –3**i** + 4**j**, find the following:

a.) -2v + 3w

We can multiply both components in v by -2 and also multiply both components in w by 3:
-2(**i** + 2**j**) + 3(–3**i** + 4**j**). Now multiply and we get: -2**i** + 4**j** – 9**i** + 12**j**. Now add the like terms to get:
–11**i** + 8**j.**

b.) v – 4w

We will take v and add this top -4 times w: **i** + 2**j** – 4(–3**i** + 4**j)** . Now multiply: **i** + 2**j** + 12**i** – 16**j)**. Now add the like terms to get: 13**i** – 14**j**.

c.) $\|v + w\|$

Inside we need to add v and w: || **i** + 2**j** + –3**i** + 4**j** ||. Now add the like terms: ||–2**i** + 6**j** ||. Now we need to find the magnitude using the formula $\|v\| = \sqrt{a^2 + b^2}$ and put in our given information: $\|v\| = \sqrt{(-2)^2 + (6)^2}$. Now square the terms: $\|v\| = \sqrt{4+36}$. We will get $\|v\| = \sqrt{40}$, so $\|v\| = 2\sqrt{10}$.

Unit Vector – a vector with a magnitude of 1.

Finding a unit vector u in the same direction as v: $u = \frac{v}{\|v\|}$

This finds a vector u that is going in the same direction as a given vector v but has a magnitude of 1.

EXAMPLE: Find a unit vector u that has the same direction as the vector v = 12**i** + 5**j**.

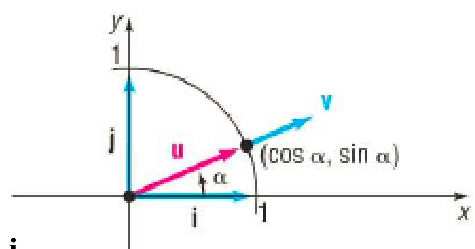
We need to first find $\|v\|$. Here a is 12 and b is 5. We will use the formula $\|v\| = \sqrt{a^2 + b^2}$ and put in our given information: $\|v\| = \sqrt{(12)^2 + (5)^2}$. Now square the terms: $\|v\| = \sqrt{144 + 25}$. We will get $\|v\| = \sqrt{169}$, so $\|v\| = 13$. Now we will use the formula $u = \frac{v}{\|v\|}$ and put in or information: $u = \frac{12\hat{i} + 5\hat{j}}{13}$. Then we can split this up to get: $u = \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$. (Some books use this notation instead of the bold letters).

EXAMPLE: Find a unit vector u that has the same direction as the vector v = –2**i** + 3**j**.

We need to first find $\|v\|$, which is $\|v\| = \sqrt{(-2)^2 + (3)^2}$. We will get $\|v\| = \sqrt{13}$. Now we will use the formula $u = \frac{v}{\|v\|}$ and put in or information: $u = \frac{-2\hat{i} + 3\hat{j}}{\sqrt{13}}$. After rationalizing we get: $u = -\frac{2\sqrt{13}}{13}\hat{i} + \frac{3\sqrt{13}}{13}\hat{j}$.

Vector components

A vector is made up of a horizontal and vertical part. These parts are called components. The picture to the right shows how to break up a vector into components if an angle is given. The formula we will use to break up into components is: $v = \|v\|\cos\alpha\,\mathbf{i} + \|v\|\sin\alpha\,\mathbf{j}$



EXAMPLE: Given $\|v\| = 8$ and $\alpha = 30^\circ$, write the vector **v** in terms of **i** and **j**.

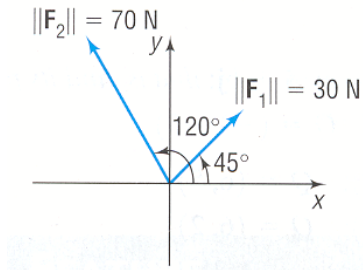
Following the formula we have $v = 8\cos 30^\circ\,\mathbf{i} + 8\sin 30^\circ\,\mathbf{j}$. Now we simplify: $v = 8 \cdot \frac{\sqrt{3}}{2}\mathbf{i} + 8 \cdot \frac{1}{2}\mathbf{j}$. So our final answer is $v = 4\sqrt{3}\,\mathbf{i} + 4\mathbf{j}$.

Resultant vector: the result when two vectors are added together.

Direction of Resultant: If the resultant vector is in the form $F = a\hat{i} + b\hat{j}$, then you can find the angle between this vector and the x axis by using the formula: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ if the resultant vector is quadrant 1 or 4. Also $\theta = \tan^{-1}\left(\frac{b}{a}\right) + 180^\circ$ if the resultant vector is in quadrant 2 or 3.

Static Equilibrium: The sum of all the forces is zero, which means nothing is moving. The magnitude and direction will both be zero in this case.

EXAMPLE: Two forces of magnitude 30 newtons (N) and 70 newtons act on an object at angle of 45 degrees and 120 degrees with the positive x-axis as shown in the figure. Find the direction and magnitude of the resultant force, that is, find $F_1 + F_2$. Then find what additional force is needed for the object to be in static equilibrium.



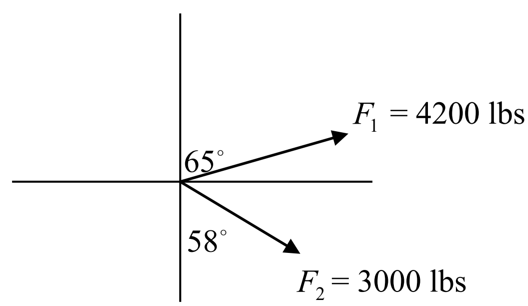
We need to break this up into components.

$$F_1 = 30\cos 45^\circ \hat{i} + 30\sin 45^\circ \hat{j} = 21.21\hat{i} + 21.21\hat{j}$$
$$F_2 = 70\cos 120^\circ \hat{i} + 70\sin 120^\circ \hat{j} = -35\hat{i} + 60.62\hat{j}$$

Now let's add these together. First add the $\hat{i}$ components and then the $\hat{j}$ components. We will get: $F_1 + F_2 = -13.79\hat{i} + 81.83\hat{j}$. To find the magnitude we use the formula: $\|F_1 + F_2\| = \sqrt{(-13.79)^2 + (81.83)^2}$. You will get: $\|F_1 + F_2\| = 82.98$. To find the direction we will use: $\theta = \tan^{-1}\left(\frac{81.83}{-13.79}\right) + 180^\circ = 99.57^\circ$. We need to add 180 degrees since our resultant $F_1 + F_2 = -13.79\hat{i} + 81.81\hat{j}$ ends up in the second quadrant.

Next we need to find what additional force is necessary to reach static equilibrium. In static equilibrium all the forces need to add to zero. Therefore, the force necessary should be $F = 13.79\hat{i} - 81.81\hat{j}$. This is because if we add this to our first resultant force, the result will be 0 for both vector components. So the easy way to find this force is to simply reverse the signs of the original resultant force.

EXAMPLE: The magnitude and direction exerted by two tugboats towing a ship are 4200 pounds at $N65^\circ E$ and 3000 pounds at $S58^\circ E$ respectively. Find the magnitude and the bearing of the resultant force.



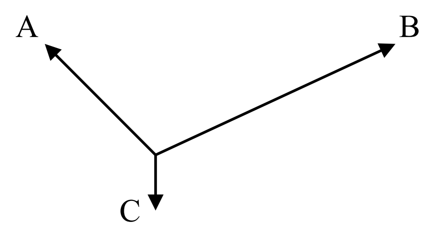
We need to break this up into components.

When you do components the angle must be measured from the x-axis. For the first force we will subtract 65 from 90 to get 25 degrees. For the second force we can subtract 58 from 90 to get 32 degrees. However, since this is below the x-axis, this angle is actually negative, so we want to use -32 degrees (or 328°).

$$F_1 = 4200\cos 25^\circ \hat{i} + 4200\sin 25^\circ \hat{j} = 3806.49\hat{i} + 1774.99\hat{j}$$
$$F_2 = 3000\cos(-32^\circ)\hat{i} + 3000\sin(-32^\circ)\hat{j} = 2544.14\hat{i} - 1589.76\hat{j}$$

Now let's add these together. First add the $\hat{i}$ components and then the $\hat{j}$ components. We will get:
$F_1 + F_2 = 6350.63\hat{i} + 185.23\hat{j}$. To find the magnitude we use the formula:
$\|F_1 + F_2\| = \sqrt{(6350.63)^2 + (185.23)^2}$. You will get: $\|F_1 + F_2\| = 6353.33$ pounds. To find the direction we will use: $\theta = \tan^{-1}\left(\dfrac{185.23}{6350.63}\right) = 1.67^\circ$. This is measured from the x-axis, so we need to subtract this from 90 degrees to get the bearing since the bearing is always measured from the vertical axis: $90 - 1.67 = 88.3^\circ$. Our bearing is $N88.3^\circ E$.

EXAMPLE: A 9.73 pound picture is hung from 2 wires as shown below. The tension on A is 3.4 pounds at 161 degrees. The tension on B is 9.2 pounds at 69.55 degrees. The tension on C is 9.73 pounds at -90 degrees. Find the magnitude and direction of the resultant force.



We will write the vectors A, B, and C in component form using the given information:

A: $3.4\cos 161^\circ \hat{i} + 3.4\sin 161^\circ \hat{j} = -3.21\hat{i} + 1.11\hat{j}$

B: $9.2\cos 69.55^\circ \hat{i} + 9.2\sin 69.55^\circ \hat{j} = 3.21\hat{i} + 8.62\hat{j}$

C: $9.73\cos(-90^\circ)\hat{i} + 9.73\sin(-90^\circ)\hat{j} = 0\hat{i} - 9.73\hat{j}$

If we add A, B, and C we get $0\hat{i} + 0\hat{j}$. This means the magnitude and direction are all 0. Therefore, we have another state where the vectors are in static equilibrium. This is something you will cover more in physics and also engineering statics courses.