9.5 The Dot Product

The dot product is one way to multiply vectors. You will get a number for an answer. In physics you will also learn the cross product in which a vector is the answer.

The Dot Product

If $u = a_1\mathbf{i} + b_1\mathbf{j}$ and $v = a_2\mathbf{i} + b_2\mathbf{j}$ then the dot product $u \cdot v = a_1 \cdot a_2 + b_1 \cdot b_2$.

EXAMPLE: Given u = –3**i** + 4**j** and v = 6**i** + 5**j**, find the following:

a.) $u \cdot v$

We will use the dot product formula. Here, $a_1 = -3$, $a_2 = 6$, $b_1 = 4$, $b_2 = 5$. Plug these into the formula: $u \cdot v = (-3)(6) + (4)(5)$. Simplifying we get $u \cdot v = 2$.

b.) $v \cdot u$

We will use the dot product formula. Here, $a_1 = 4$, $a_2 = 5$, $b_1 = -3$, $b_2 = 6$. Plug these into the formula: $v \cdot u = (4)(5) + (-3)(6)$. Simplifying we get $v \cdot u = 2$. Notice that it doesn't matter what order we multiply these vectors when using the dot product. We get the same answer.

c.) $u \cdot u$

We will use the dot product formula. Here, $a_1 = -3$, $a_2 = -3$, $b_1 = 4$, $b_2 = 4$. Plug these into the formula: $u \cdot u = (-3)(-3) + (4)(4)$. Simplifying we get $u \cdot u = 25$.

d.) $v \cdot v$

We will use the dot product formula. Here, $a_1 = 6$, $a_2 = 6$, $b_1 = 5$, $b_2 = 5$. Plug these into the formula: $v \cdot v = (6)(6) + (5)(5)$. Simplifying we get $v \cdot v = 61$.

e.) $\|u\|$

We will use the formula $\|u\| = \sqrt{a^2 + b^2}$: $\|u\| = \sqrt{(-3)^2 + (4)^2}$ = 5.

f.) $\|v\|$

We will use the formula $\|v\| = \sqrt{a^2 + b^2}$: $\|v\| = \sqrt{(6)^2 + (5)^2} = \sqrt{61}$.

Notice that if you multiply and a vector by itself using the dot product this is equal to magnitude squared. This is a property: $u \cdot u = \|u\|^2$. Also, $v \cdot v = \|v\|^2$. Another property is $u \cdot v = v \cdot u$.

Angle between two vectors

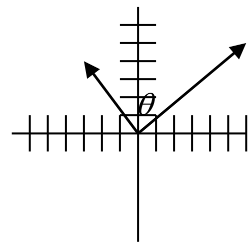
Given two vectors u and v, the angle between the two vectors is given as:

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}$$

EXAMPLE: u = –3**i** + 4**j** and v = 6**i** + 5**j**, find the angle between u and v.

We already know from the previous problem that $u \cdot v = 2$. We also know $\|u\| = 5$ and $\|v\| = \sqrt{61}$. We can put these into our formula to find the angle: $\cos\theta = \dfrac{2}{5 \cdot \sqrt{61}}$.

Putting this into a calculator will give: $\cos\theta = 0.0512$, so after taking the inverse cosine we will get our answer: $\theta = 87.06^{\circ}$. In the picture to the right we drew the vectors and you can see the angle between them looks like it's about 87 degrees.



If the angle between the vectors is 0 or 180 degrees, then the vectors are **parallel**.

If the angle between the vectors is 90 degrees then the vectors are **orthogonal**.

EXAMPLE: Given u = –3**i** + 2**j** and v = 4**i** + 6**j**, find the following:

a.) $u \cdot v$

We will use the dot product formula. Here, $a_1 = -3$, $a_2 = 4$, $b_1 = 2$, $b_2 = 6$. Plug these into the formula: $u \cdot v = (-3)(4) + (2)(6)$. Simplifying we get $u \cdot v = 0$.

b.) $\|u\|$

We will use the formula $\|u\| = \sqrt{a^2 + b^2}$: $\|u\| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$.

c.) $\|v\|$

We will use the formula $\|v\| = \sqrt{a^2 + b^2}$: $\|v\| = \sqrt{(4)^2 + (6)^2} = \sqrt{52}$.

d.) The angle between u and v.

We will use the formula $\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|}$ and plug in our known values: $\cos\theta = \dfrac{0}{\sqrt{13}\sqrt{52}}$. So $\cos\theta = 0$. After

taking the inverse cosine we get $\theta = 90^{\circ}$, so this means that θ is orthogonal.

EXAMPLE: Are the vectors u = 2**i** + 5**j** and v = 3**i** – 7**j** orthogonal, parallel, or neither?

We need to use the formula $\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}$. First we will find $u \cdot v = 2(3) + 5(-7) = -29$. Then we can find $\|u\|$ by using the formula: $\|u\| = \sqrt{(2)^2 + (5)^2} = \sqrt{29}$. Then we can find $\|v\|$ by using the formula: $\|v\| = \sqrt{(3)^2 + (-7)^2} = \sqrt{58}$. Now we can put these into the formula to find our angle: $\cos\theta = \frac{-29}{\sqrt{29}\sqrt{58}}$. We can put this into our calculator to get: $\cos\theta = -0.7071$. So $\theta = 135^\circ$. This is not orthogonal or parallel, so we will answer **neither**.