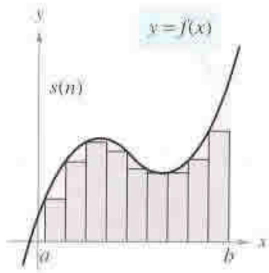
5.2 Approximating Areas with Limits of Finite Sums

This chapter deals with finding the area under curves, which is what integrals will do. In this section, we will be finding the area of rectangular strips, and then adding all of these together. There are two ways to add rectangles together:

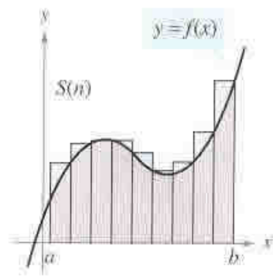
Upper and Lower Sums

The diagrams below explain the difference between inscribed and circumscribed rectangles. It also shows the difference between the upper and lower sum. In the drawing S(n) represents the sum of the individual areas.



Area of inscribed rectangles is less than area of region.

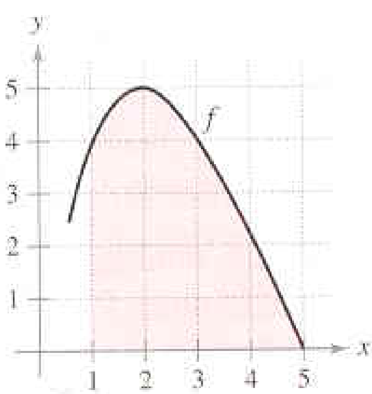
Lower Sum



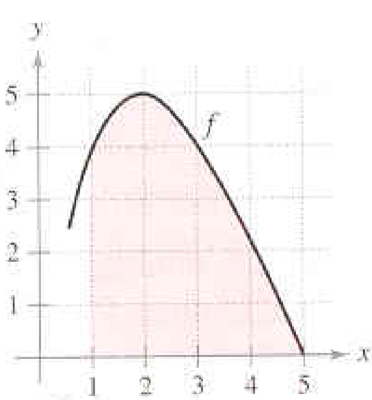
Area of circumscribed rectangles is greater than area of region.

Upper Sum

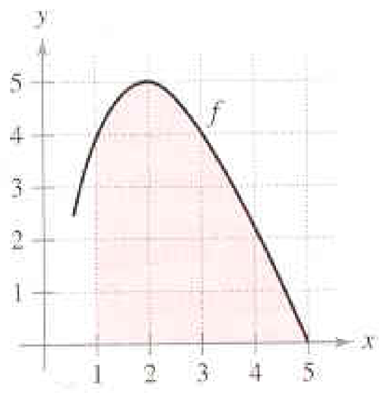
EXAMPLE: Estimate the area under the curve on [1, 5] by using upper and lower sums. Use rectangles of width 1.



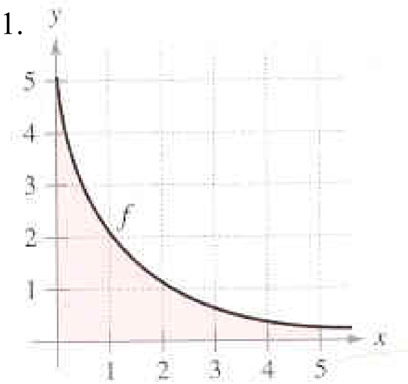
Lower Sums



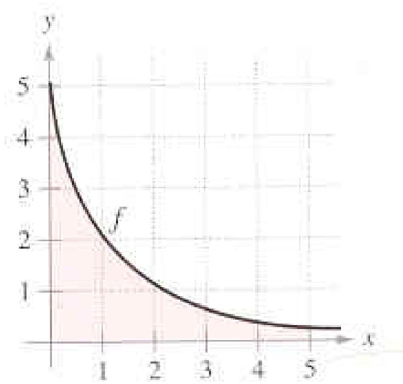
Upper Sums



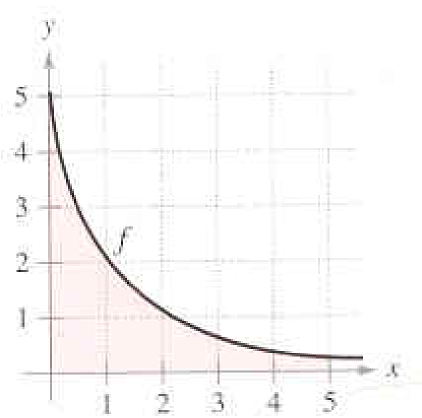
EXAMPLE: Estimate the area under the curve on [0, 5] by using upper and lower sums. Use rectangles of width 1.



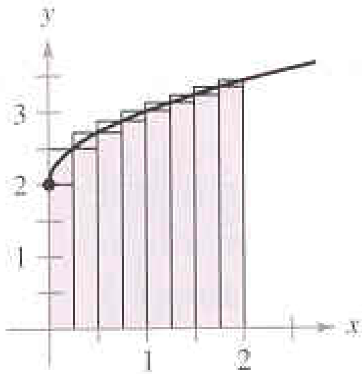
Lower Sums



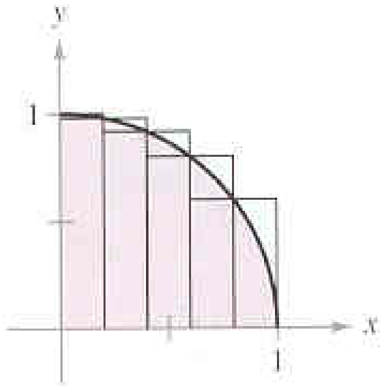
Upper Sums



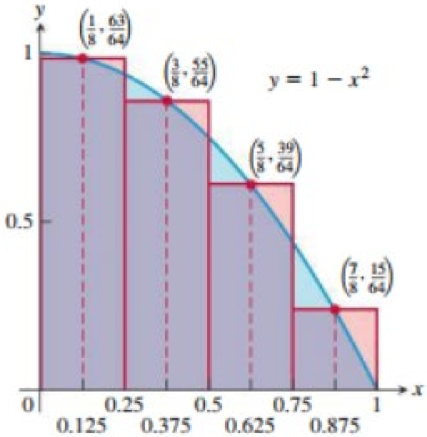
EXAMPLE: Use upper and lower sums to approximate the area under the curve $y = \sqrt{x} + 2$ and above the x-axis on the interval [0, 2]. Use 8 subintervals as shown below:



EXAMPLE: Use upper and lower sums to approximate the area under the curve $y=\sqrt{1-x^2}$ and above the x-axis on the interval [0, 1]. Use 5 subintervals as shown below:



EXAMPLE: Use rectangles each of whose height is determined by $y = 1 - x^2$ at the midpoint of the rectangle's base (*the midpoint rule*) and estimate the area under the curve $y = 1 - x^2$ by using four rectangles between the x-values of 0 and 1.



Sigma Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_n$$

EXAMPLE: Evaluate: $\sum_{i=1}^{4} 3i - 2$.

EXAMPLE: Evaluate: $\sum_{k=1}^{4} (-1)^k \cos(k\pi)$.

EXAMPLE: Use sigma notation to write the sum: $\frac{5}{1+1} + \frac{5}{1+2} + \frac{5}{1+3} + ... + \frac{5}{1+15}$.

EXAMPLE: Use sigma notation to write the sum: $-\frac{1}{7} + \frac{2}{7} - \frac{3}{7} + \frac{4}{7} - \frac{5}{7} + \frac{6}{7} - 1$.

Algebra Rules for Finite Sums

1.) $\sum_{i=1}^{n} k \cdot a_i = k\sum_{i=1}^{n} a_i$ Any constants we can write in front of the summation symbol.

2.) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ We can split up the summation into two separate ones.

Summation Formulas

1.) $\sum_{i=1}^{n} c = c \cdot n$

2.) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

3.) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$

4.) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$

EXAMPLE: Evaluate: $\sum_{i=1}^{15} 2i-3$ using properties of summation and the summation formulas.

EXAMPLE: Evaluate: $\sum_{i=1}^{10} i^2 - 2i^3$ using properties of summation and the summation formulas.

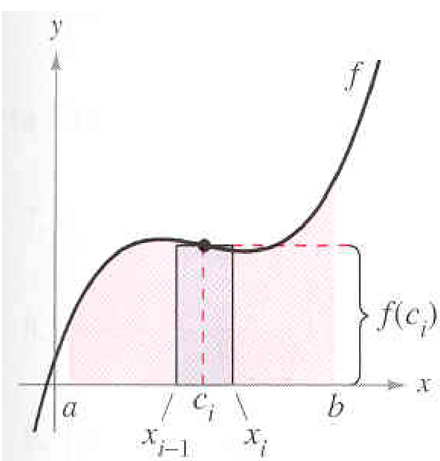
EXAMPLE: Find the limit: $\lim_{n\to\infty} \frac{1}{n^2}\left[\sum_{i=1}^{n} i\right]$.

EXAMPLE: Find the limit: $\lim_{n\to\infty} \frac{64}{n^3}\left[\sum_{i=1}^{n} i^2\right]$.

EXAMPLE: Find the limit: $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$.

Using the Limit Process to Find the Area Under a Curve

In order to find the area under a curve, we are going to find the area of several rectangles and then add them all together to get the total area. Let's first look at one of those rectangles and define some variables:



EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into n equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve $f(x) = 1 + x^2$ over [0, 3].

EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into n equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve $f(x) = 5x$ over [1, 3].

EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into n equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve $f(x) = x + x^3$ over [0, 1].