

5.5 Substitution with Indefinite Integrals

Integration by Substitution:

Let g be a function whose range is in an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then:

$$\int f(g(x)) \cdot g'(x) dx = F(g(x)) + C.$$

$$\text{If } u = g(x) \text{ then } du = g'(x) dx \text{ and } \int f(u) du = F(u) + C.$$

Like in the chain rule, $g(x)$ is usually an 'inside' function that needs to be identified

How to Integrate by Substitution:

- 1.) Let $u = g(x)$.
- 2.) Take the derivative of both sides to get $du = g'(x) dx$.
- 3.) Solve for dx and substitute this and u into the equation.
- 4.) Take the antiderivative of u .
- 5.) Substitute back in the $g(x)$ for u .

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form:

$$\int (x^2 - 9)^3 (2x) \, dx, \quad u = x^2 - 9.$$

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form:

$$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) \, dx, \quad u = \frac{1}{x}.$$

EXAMPLE: Integrate by substitution: $\int \frac{x^2}{\sqrt{16-x^3}} dx$.

EXAMPLE: Integrate by substitution: $\int x^3 \sqrt{1 + \frac{x^4}{8}} dx$.

EXAMPLE: Integrate by substitution: $\int \frac{\sin(2x)}{\cos^3(2x)} dx$.

EXAMPLE: Integrate by substitution: $\int \sec^2(1-x) \tan^7(1-x) dx$.

EXAMPLE: Integrate by substitution: $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2} - 7\right) dx$.

Integrals Involving Inverse Trigonometric Functions with Substitutions

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{|u|}{a} + C$$

EXAMPLE: Integrate by substitution: $\int \frac{4}{1+9x^2} dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{dx}{x^2 + 4x + 13}$.

EXAMPLE: Integrate by substitution: $\int \frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^3}{\sqrt{9-x^2}} dx$.

Integration of a Natural Logarithm

Let u be a differentiable function of x . Then:

$$1.) \int \frac{1}{x} dx = \ln|x| + C$$

$$2.) \int \frac{1}{u} du = \ln|u| + C \quad \text{Since } du = u' dx \text{ we can rewrite this as: } \int \frac{u'}{u} du = \ln|u| + C$$

EXAMPLE: Find the indefinite integral: $\int \frac{x(x+2)}{x^3 + 3x^2 - 4} dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{3 \sec^2 t}{6 + 3 \tan t} dt$.

EXAMPLE: Find the indefinite integral: $\int \frac{1}{x^{\frac{2}{3}} \left(1 + x^{\frac{1}{3}}\right)} dx$.

EXAMPLE: Find the indefinite integral: $\int \tan x \, dx$.

Integrals of the Six Trigonometric Functions

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE: Find the indefinite integral: $\int \sec \frac{x}{2} dx$.

EXAMPLE: Find the indefinite integral: $\int \csc \theta + \cot \theta d\theta$.

EXAMPLE: Find the indefinite integral: $\int (\sin 2\theta)e^{\cos^2 \theta} d\theta$.

Change of Variables

In all the problems we did we were able to always cancel out the x terms and just have u . In the next two problems this will not automatically happen so we need to change variables.

EXAMPLE: Integrate by substitution: $\int x(1-x)^4 dx$.

EXAMPLE: Integrate by substitution: $\int \frac{2x+1}{\sqrt{x+4}} dx$.