5.5 Substitution with Indefinite Integrals

Integration by Substitution:

Let g be a function whose range is in an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then:

$\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C$.

If $u = g(x)$ then $du = g'(x)dx$ and $\int f(u)du = F(u) + C$.

Like in the chain rule, $g(x)$ is usually an 'inside' function that needs to be identified

How to Integrate by Substitution:

1.) Let $u = g(x)$.

2.) Take the derivative of both sides to get $du = g'(x)\ dx$.

3.) Solve for dx and substitute this and u into the equation.

4.) Take the antiderivative of u.

5.) Substitute back in the $g(x)$ for u.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form: $\int (x^2 - 9)^3 (2x)\ dx$, $u = x^2 - 9$.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form: $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right)\ dx$, $u = \frac{1}{x}$.

EXAMPLE: Integrate by substitution: $\int \frac{x^2}{\sqrt{16-x^3}}\,dx$.

EXAMPLE: Integrate by substitution: $\int x^3\sqrt{1+\frac{x^4}{8}}\,dx$.

EXAMPLE: Integrate by substitution: $\int \frac{\sin(2x)}{\cos^3(2x)}\ dx$.

EXAMPLE: Integrate by substitution: $\int \sec^2(1-x)\tan^7(1-x)\ dx$.

EXAMPLE: Integrate by substitution: $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}-7\right)\,dx$.

Integrals Involving Inverse Trigonometric Functions with Substitutions

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a}+C$$

EXAMPLE: Integrate by substitution: $\int \frac{4}{1+9x^2}\,dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{dx}{x^2+4x+13}$.

EXAMPLE: Integrate by substitution: $\int \frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^3}{\sqrt{9-x^2}}\, dx$.

Integration of a Natural Logarithm

Let u be a differentiable function of x. Then:

1.) $\int \frac{1}{x}\, dx = \ln|x| + C$

2.) $\int \frac{1}{u}\, du = \ln|u| + C$ Since $du = u'\, dx$ we can rewrite this as: $\int \frac{u'}{u}\, du = \ln|u| + C$

EXAMPLE: Find the indefinite integral: $\int \frac{x(x+2)}{x^3 + 3x^2 - 4}\, dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{3\sec^2 t}{6 + 3\tan t}\, dt$.

EXAMPLE: Find the indefinite integral: $\int \frac{1}{x^{\frac{2}{3}}\left(1+x^{\frac{1}{3}}\right)}\,dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{2x^2+7x-3}{x-2}\,dx$.

EXAMPLE: Find the indefinite integral: $\int \frac{3x^3 - x^2 + x - 2}{x^2 + 2}\ dx$.

EXAMPLE: Find the indefinite integral: $\int \tan x\ dx$.

Integrals of the Six Trigonometric Functions

$\int \sin u \;\, du = -\cos u + C$

$\int \cos u \;\, du = \sin u + C$

$\int \tan u \;\, du = -\ln\left|\cos u\right| + C$

$\int \cot u \;\, du = \ln\left|\sin u\right| + C$

$\int \sec u \;\, du = \ln\left|\sec u + \tan u\right| + C$

$\int \csc u \;\, du = -\ln\left|\csc u + \cot u\right| + C$

EXAMPLE: Find the indefinite integral: $\int \sec \frac{x}{2} \;\, dx$.

EXAMPLE: Find the indefinite integral: $\int \csc\theta + \cot\theta \;\, d\theta$.

EXAMPLE: Find the indefinite integral: $\int (\sin 2\theta) e^{\cos^2 \theta} \, d\theta$.

Change of Variables

In all the problems we did we were able to always cancel out the x terms and just have u. In the next two problems this will not automatically happen so we need to change variables.

EXAMPLE: Integrate by substitution: $\int x(1-x)^4 \, dx$.

EXAMPLE: Integrate by substitution: $\int \frac{2x+1}{\sqrt{x+4}}\, dx$.