1.1 Review of Algebra and Factoring

In this section we will review various algebra tools necessary for Calculus.

EXAMPLE: Expand: $(x-10y)^2$.

The first thing to recognize is that $(x-10y)^2 \neq x^2 - 10^2 y^2$. We cannot distribute the exponent. Instead, let's rewrite this as: $(x-10y)^2 = (x-10y)(x-10y)$. Now we can use the FOIL method:

First: $x \cdot x = x^2$
Outer: $x(-10y) = -10xy$
Inner: $-10y \cdot x = -10xy$
Last: $-10y(-10)y = 100y^2$

For the final answer, combine all like terms and arrange the terms in descending powers of x.
$(x-10y)^2 = x^2 - 10xy - 10xy + 100y^2 = x^2 - 20xy + 100y^2$.

Now we will take a look at factoring, which is the opposite of FOILing.

EXAMPLE: Factor $63x^3 + 21x^2$. Factor out –1 if the leading coefficient is negative.

First we will find the Greatest Common Factor (GCF) of each term:

$63x^3 = 3 \cdot 3 \cdot 7 \cdot x \cdot x \cdot x$
$21x^2 = 3 \cdot 7 \cdot x \cdot x$

Now we want to find any factors that appear in both factorizations. We see that $3 \cdot 7 \cdot x \cdot x = 21x^2$ appears in both. Now we will factor out the $21x^2$: $63x^3 + 21x^2 = 21x^2(\quad + \quad)$. To find out what goes inside the parenthesis, divide each term by the GCF: : $\dfrac{63x^3}{21x^2} = 3x$ and $\dfrac{21x^2}{21x^2} = 1$. Therefore, $63x^3 + 21x^2 = 21x^2(3x+1)$.

Difference of Squares Factoring Formula: $a^2 - b^2 = (a+b)(a-b)$

EXAMPLE: Factor using difference of squares: $4x^2 - 9$

First we want to rewrite the original problem as squares: $4x^2 - 9 = 2^2x^2 - 3^2 = (2x)^2 - 3^2$. Now we know that $a = 2x$ and $b = 3$. We will put this into the difference of squares formula to get our answer:
$4x^2 - 9 = (2x+3)(2x-3)$.

EXAMPLE: Factor $x^2 - 10x - 24$ completely. If the polynomial cannot be factored, say it is prime.

For these kinds of problems in which there is a 1 in front of the x^2, we are looking for two numbers that multiply to make –24 and add to be –10. For this, one factor should be positive and the other negative.

1.) First let's write all the pairs of numbers that multiply to make –24:
   1,–24   –1,24   2,–12   –2,12   3,–8   –3,8   4,–6   –4,6
2.) We see that 2 and –12 will work since they multiply to make –24 but add together to make –10.
3.) We will write the factored expression as: $x^2 - 10x - 24 = (x+2)(x-12)$.

EXAMPLE: Factor $7x^3 - 28x^2 - 315x$ completely. If the polynomial cannot be factored, say it is prime.

We begin this problem by factoring out the common factor, 7*x*: $7x(x^2 - 4x - 45)$. Next we will factor what is inside the parenthesis. We will get: $7x(x+5)(x-9)$. There is no more factoring that can be done, so we have factored it completely. Therefore, $7x^3 - 28x^2 - 315x = 7x(x+5)(x-9)$.

EXAMPLE: Factor $6x^2 + 23x + 20$ using the grouping method, if possible.

For problem in which the number in front of the x^2 is not a 1, we need to use a different method. I will show the Grouping Method here, which is also called the AC Method. The steps are indicated below.

1.) First, we will multiply $a \cdot c = 6 \cdot 20 = 120$.
2.) Now let's write all the pairs of numbers that multiply to make 120. These can all be positive since the middle term is positive:
   1,120   2,60   3,40   4,30   5,24   6,20   8,15   10,12
3.) We see that 8 and 15 will work since they multiply to make 120 but add together to make 23.
4.) We will rewrite the original expression as: $6x^2 + 8x + 15x + 20$. It does not matter which factor comes first.
5.) Now we need to factor this using the grouping method:

$$\begin{aligned} 6x^2 + 8x + 15x + 20 &= (6x^2 + 8x) + (15x + 20) \\ &= 2x(3x+4) + 5(3x+4) \\ &= (3x+4)(2x+5) \end{aligned}$$

EXAMPLE: Factor $2x^2 + 9x - 35$ using the grouping method, if possible.

1.) First, we will multiply $a \cdot c = 2 \cdot -35 = -70$.
2.) Now let's write all the pairs of numbers that multiply to make -70. One factor will be positive and another one will be negative:
   –1, 70   1, –70   –2, 35   2, –35   –5, 14   5, –14   –7, 10   7, –10
3.) We see that –5 and 14 will work since they multiply to make –70 but add together to make 9.
4.) We will rewrite the original expression as: $2x^2 - 5x + 14x + 35$. It does not matter which factor comes first.
5.) Now we need to factor this using the grouping method:

$$\begin{aligned} 2x^2 - 5x + 14x + 35 &= (2x^2 - 5x) + (14x + 35) \\ &= x(2x-5) + 7(2x-5) \\ &= (2x-5)(x+7) \end{aligned}$$

Factoring out common factors with fractional exponents

EXAMPLE: Factor completely: $x^{\frac{7}{2}} + 8x^{\frac{5}{2}} + 12x^{\frac{3}{2}}$

Look for the smallest fraction, and that is what you factor out. Here we factor out $x^{\frac{3}{2}}$. Factoring is the same as division. We are really dividing each monomial by $x^{\frac{3}{2}}$. Here is what it looks like: $\dfrac{x^{\frac{7}{2}}}{x^{\frac{3}{2}}} + \dfrac{8x^{\frac{5}{2}}}{x^{\frac{3}{2}}} + \dfrac{12x^{\frac{3}{2}}}{x^{\frac{3}{2}}}$

When you divide you subtract the exponents. Once you subtract you will get $x^2 + 8x + 12$. Don't forget to write the number that you factored out, so now you have:

$x^{\frac{3}{2}}\left(x^2 + 8x + 12\right)$. We are not done because it is not fully factored. So we factor the trinomial.

$x^{\frac{3}{2}}(x+2)(x+6)$ Now we are done.

EXAMPLE: Solve: $6x^{\frac{7}{3}} - x^{\frac{4}{3}} - 15x^{\frac{1}{3}} = 0$

In order to solve this, we must first factor. Look for the smallest fraction, and that is what you factor out. Here we factor out $x^{\frac{1}{3}}$. Factoring is the same as division. We are really dividing each monomial by $x^{\frac{1}{3}}$. Here is what it looks like: $\dfrac{6x^{\frac{7}{3}}}{x^{\frac{1}{3}}} - \dfrac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}} - \dfrac{15x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

When you divide you subtract the exponents. Once you subtract you will get $6x^2 - x - 15$. Don't forget to write the number that you factored out, so now you have:

$x^{\frac{1}{3}}\left(6x^2 - x - 15\right)$. We are not done because it is not fully factored. So we factor the trinomial.

$x^{\frac{1}{3}}(2x+3)(3x-5)$ Now we will set each factor equal to zero and solve: $x^{\frac{1}{3}} = 0$, $2x+3=0$, $3x-5=0$.

Solving each of these will give us the answers of $x = 0, -\frac{3}{2}, \frac{5}{3}$.

Radical expressions can be written without the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index *n* is even, then *a* cannot be negative. Below are formulas for converting a rational exponent to a radical.

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\qquad$ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

EXAMPLE: Convert the radical expression to an equivalent expression involving rational exponents: $\sqrt[9]{x^8}$.

In this problem the index *n* is 9, the *a* is x, and *m* is 8, so $\sqrt[9]{x^8} = x^{\frac{8}{9}}$.

Laws of Exponents

$a^s \cdot a^t = a^{s+t}$ Example: $2^5 \cdot 2^3 = 2^{5+3} = 2^8$

$\frac{a^s}{a^t} = a^{s-t}$ Example: $\frac{2^6}{2^3} = 2^{6-3} = 2^3$

$(a^s)^t = a^{s \cdot t}$ Example: $(2^3)^5 = 2^{3 \cdot 5} = 2^{15}$

$(a \cdot b)^s = a^s \cdot b^s$ Example: $(2 \cdot 3)^5 = 2^5 3^5$

$1^s = 1$ Example: $1^{34} = 1$

$a^0 = 1$ Example: $4^0 = 1$, $\pi^0 = 1$

$a^{-s} = \frac{1}{a^s}$ Example: $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$\left(\frac{a}{b}\right)^{-s} = \left(\frac{b}{a}\right)^s$ Example: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

EXAMPLE: Convert the radical expression to an equivalent expression involving negative rational exponents: $\frac{1}{\sqrt[9]{x^{11}}}$.

First we want to rewrite the expression in the denominator using exponents: $\frac{1}{\sqrt[9]{x^{11}}} = \frac{1}{x^{11/9}}$. When we move the x in the denominator to the top of the fraction, the exponent becomes negative: $\frac{1}{x^{11/9}} = x^{-\frac{11}{9}}$

EXAMPLE: Simplify $\frac{8x^3 - x^2 + 3\sqrt{x}}{x^3}$ and write with positive exponents.

Divide everything in the numerator separately by the denominator.

$\frac{8x^3 - x^2 + 3x^{\frac{1}{2}}}{x^3} = \frac{8x^3}{x^3} - \frac{x^2}{x^3} + \frac{3x^{\frac{1}{2}}}{x^3}$. Next we want to subtract the exponents.

$\frac{8x^3}{x^3} - \frac{x^2}{x^3} + \frac{3x^{\frac{1}{2}}}{x^3} = 8x^{3-3} - x^{2-3} + x^{\frac{1}{2}-3} = 8x^0 - x^{-1} + x^{-\frac{5}{2}}$. Notice that we have negative exponents. If we put this in the denominator, then it will become positive. Also, $x^0 = 1$: $8x^0 - x^{-1} + x^{-\frac{5}{2}} = 8 - \frac{1}{x} + \frac{3}{x^{\frac{5}{2}}}$