

# 1.4 Inverse Functions

## Inverse Function:

Notation to write “the inverse of  $f(x)$ ” is  $f^{-1}(x)$ . If two functions  $f$  and  $f^{-1}$  are inverses then the following must be true:  $(f \circ f^{-1})(x) = x$  and  $(f^{-1} \circ f)(x) = x$ . Note that these functions must be one-to-one.

EXAMPLE: Given  $f(x) = 2x - 1$  and  $f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$  verify that they are inverses.

We need to show  $(f \circ f^{-1})(x) = x$  and  $(f^{-1} \circ f)(x) = x$ . Let's first find  $(f \circ f^{-1})(x)$ .

$(f \circ f^{-1})(x) = f(f^{-1}(x))$  Start with the definition.

$(f \circ f^{-1})(x) = f\left(\frac{1}{2}x + \frac{1}{2}\right)$  We remove the  $f^{-1}(x)$  and replace it with  $\frac{1}{2}x + \frac{1}{2}$ .

$(f \circ f^{-1})(x) = 2\left(\frac{1}{2}x + \frac{1}{2}\right) - 1$  Now simplify.

$$(f \circ f^{-1})(x) = x + 1 - 1$$

$(f \circ f^{-1})(x) = x$  We have shown this is true. Now we need to show  $(f^{-1} \circ f)(x) = x$ .

$(f^{-1} \circ f)(x) = f^{-1}(f(x))$  First start with the definition.

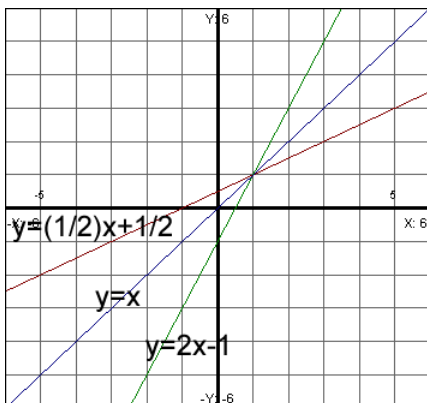
$(f^{-1} \circ f)(x) = f^{-1}(2x - 1)$  We remove the  $f(x)$  and replace it with  $2x - 1$ .

$(f^{-1} \circ f)(x) = \frac{1}{2}(2x - 1) + \frac{1}{2}$  Now simplify.

$$(f^{-1} \circ f)(x) = x - \frac{1}{2} + \frac{1}{2}$$

$(f^{-1} \circ f)(x) = x$  We have shown this is true. So we have verified they are inverses.

What is the significance of the  $x$ ? Why do we get  $x$  when we simplify? I'm glad you asked! Let's look at the graph of  $f$  and  $f^{-1}$ . I will also graph  $y = x$ . Notice that  $f$  and  $f^{-1}$  are symmetric to the line  $y = x$ . This is always the case with inverses. Notice also that points on the graph of  $f(x)$  are reversed on  $f^{-1}(x)$ . For example, on the  $f(x)$  line we see the points  $(2, 3)$  and  $(-1, -3)$ . On the graph of  $f^{-1}(x)$  we get the points  $(3, 2)$  and  $(-3, -1)$ .



**How to find an inverse algebraically:**

Step 1: Replace  $f(x)$  with  $y$ .

Step 2: Switch  $x$  and  $y$ .

Step 3: Solve for  $y$ .

Step 4: Replace  $y$  with  $f^{-1}(x)$ .

EXAMPLE: Given  $f(x) = \sqrt{x+7}$  find  $f^{-1}(x)$ .

Step 1:  $y = \sqrt{x+7}$

Step 2:  $x = \sqrt{y+7}$

Step 3:  $(x)^2 = (\sqrt{y+7})^2$

$$x^2 = y + 7$$

$$x^2 - 7 = y$$

Step 4:  $x^2 - 7 = f^{-1}(x)$

Note that the domain of  $f(x)$  is  $[-7, \infty)$  and the range of  $f(x)$  is  $[0, \infty)$ . If you switch these you will get the domain and range for the inverse. The domain of  $f^{-1}(x)$  is  $[0, \infty)$  and the range of  $f^{-1}(x)$  is  $[-7, \infty)$ . So we need to restrict the domain on our answer.  $f^{-1}(x) = x^2 - 7$  where  $x > 0$ .

**Inverse Trigonometric Functions**

From trigonometry we know that  $\sin 30^\circ = \frac{1}{2}$ . We put in an angle and get a value as a result. In inverse trig functions we put in the value and get an angle:  $\sin^{-1} \frac{1}{2} = 30^\circ$ . So here we put in the value of one half and got 30 degrees as a result. We are not allowed to put any number into our inverse trig functions. There are restrictions on the domain that are given in the following table:

	Domain	Range
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

We are not going to look at the graphs of the inverse trig functions in this class, however that is where the domain and range is derived from.

EXAMPLE: Find  $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ .

What this is really asking is: “find an angle between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  that has a value of  $\frac{\sqrt{3}}{2}$ .” You will need to remember your unit circle or table for this one. This corresponds to an angle of 60 degrees, or  $\frac{\pi}{3}$ , which is the answer.

EXAMPLE: Find  $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ .

What this is really asking is: “find an angle between 0 and  $\pi$  that has a value of  $\frac{1}{\sqrt{2}}$ .” If you rationalize, then  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ . This corresponds to an angle of 45 degrees, or  $\frac{\pi}{4}$ , which is the answer.

EXAMPLE: Find  $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ .

What this is really asking is: “find an angle between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  that has a value of  $\frac{\sqrt{3}}{3}$ .” You will need to remember your unit circle or table for this one. This corresponds to an angle of 30 degrees.

EXAMPLE: Find  $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$ .

From our earlier example, we know that  $\sin^{-1}\frac{1}{2} = 30^\circ$ . So then our problem becomes  $\cos(30^\circ)$ . From the unit circle,  $\cos 30^\circ = \frac{\sqrt{3}}{2}$ . Therefore,  $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$ .

EXAMPLE: Find the exact value:  $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$ .

The unit circle tells us that  $\cos\frac{\pi}{3} = \frac{1}{2}$ . So our problem becomes  $\cos^{-1}\left(\frac{1}{2}\right)$ . We can look at the unit circle for this value, and we get  $\frac{\pi}{3}$ . There is also a property  $\cos^{-1}(\cos x) = x$  as long as  $0 \leq x \leq \pi$ , so either of these processes will allow us to find the answer,. Therefore,  $\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$ .