1.5 Exponential and Logarithmic Functions

Logarithms

In the last section we looked at inverses. In order to find an inverse we need to switch *x* and *y*. Suppose we wanted to find the inverse of our exponent function, $y = b^x$. First we need to switch *x* and *y*. We will get $x = b^y$. How do we solve for *y*? This is where we need logarithms, which are a way to solve for an exponent.

With logarithms there are two forms: **Exponential form**: $x = b^y$ **Logarithmic form**: $y = \log_b x$

EXAMPLE: Change $\log_c 6 = 8$ into exponential form.

Here b = c, y = 8 and x = 6. If we put these into the exponential form we get $c^8 = 6$.

EXAMPLE: Change $2^d = 8$ into logarithmic form.

Here x = 8, b = 2 and y = d. If we put these into the logarithmic form we get $d = \log_2 8$.

Properties of Logarithms

1.) $\log_b 1 = 0$ Example: $\log_3 1 = 0$, $\ln 1 = 0$

2.) $\log_b b = 1$ Example: $\log_2 2 = 1$, $\ln e = 1$, $\log 10 = 1$

3.) $b^{\log_b M} = M$ Example; $2^{\log_2 5} = 5$, $5^{\log_5 \pi} = \pi$

4.) $\log_b b^r = r$ Example: $\log_3 3^7 = 7$, $\log_2 2^5 = 5$

5.) $\log_b M^r = r \cdot \log_b M$ Example: $\log_3 5^8 = 8 \cdot \log_3 5$

6.) $\log_b (M \cdot N) = \log_b M + \log_b N$ Example: $\log_2 (3 \cdot 5) = \log_2 3 + \log_2 5$

7.) $\log_b \left(\frac{M}{N} \right) = \log_b M - \log_b N$ Example: $\log_3 \left(\frac{24}{6} \right) = \log_3 24 - \log_3 6$

EXAMPLE: Find the exact value using logarithm properties: $\log_3 (\log_7 21 - \log_7 3)$.

First we will use property #7 inside the parenthesis: $\log_3 \left(\log_7 \frac{21}{3} \right)$. This simplifies to $\log_3 (\log_7 7)$. Now we can use property #2 that says $\log_7 7 = 1$. Our problem now becomes $\log_3 1$. This is equal to zero according to property #1. Therefore, $\log_3 (\log_7 21 - \log_7 3) = 0$.

EXAMPLE: Find the exact value using logarithm properties: $\log_{144} 12$.

First we will rewrite this one so that the number after the log is the same as the base: $\log_{144}\sqrt{144}$. This can be written as $\log_{144} 144^{\frac{1}{2}}$. Then we will use log property #4: $\frac{1}{2}\log_{144} 144$. Since $\log_{144} 144 = 1$, our answer is $\frac{1}{2}$.

EXAMPLE: Express $\log_9 x^2 \cdot \sqrt{3x-5}$ as a sum or difference of logarithms. Express powers as factors.

Apply property #6. You will get $\log_9 x^2 + \log_9 \sqrt{3x-5}$. The square root can be written as a ½ power: $\log_9 x^2 + \log_9 (3x-5)^{\frac{1}{2}}$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $2 \cdot \log_9 x + \frac{1}{2} \cdot \log_9 (3x-5)$. This is our answer.

EXAMPLE: Express $\ln \frac{(x+5)^4}{x^3}$ as a sum or difference of logarithms. Express powers as factors.

Apply property #7. You will get $\ln(x+5)^4 - \ln x^3$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $4 \cdot \ln(x+5) - 3\ln x$.

EXAMPLE: Express $\log_4 \frac{(x-5)^5 \cdot \sqrt[3]{x-2}}{(x-1)^4}$ as a sum or difference of logarithms. Express powers as factors.

For this one you will apply both property #6 and #7. You will get $\log_4 (x-5)^5 \cdot \sqrt[3]{x-2} - \log_4 (x-1)^4$. Now we can use property #6 to break up the first log. You will get: $\log_4 (x-5)^5 + \log_4 \sqrt[3]{x-2} - \log_4 (x-1)^4$. We can rewrite the cube root as a 1/3 power: $\log_4 (x-5)^5 + \log_4 (x-2)^{\frac{1}{3}} - \log_4 (x-1)^4$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors:
$5\log_4 (x-5) + \frac{1}{3}\log_4 (x-2) - 4 \cdot \log_4 (x-1)$.

Solving Exponential Equations

Equal Bases Property (The Equivalence Property of Exponential Expressions)

If $a^u = a^v$ then $u = v$.

EXAMPLE: Solve $2^{x-2} = 8$

First we will rewrite the right side as a base of 2: $2^{x-2} = 2^3$. Since the bases are now equal, we can set the exponents equal using the equal bases property: $x - 2 = 3 \quad \Rightarrow \quad x = 5$.

EXAMPLE: Solve: $4^{x-2} - 64 = 0$.

First we isolate the exponential term: $4^{x-2} = 64$. In order to solve this, we must make both the bases the same. Since there is a 4 on the left hand side, I want to write 64 as 4 raised to some power. It is known that $4^3 = 64$ so we can now rewrite our equation:

$4^{x-2} = 4^3$. The Equivalence Property of Exponential Expressions states that if the bases are the same then we can set the exponents equal to each other. If we do this we will have x – 2 = 3. Solving this we get x = 5.

EXAMPLE: Solve: $3^x = 7$.

For this problem, we can't use the Equal Bases Property since one base can't be written in terms of the other. To solve this we will take either the natural log or regular log of both sides:

$\ln 3^x = \ln 7$	Now use property #5 to bring down the x.
$x \ln 3 = \ln 7$	Divide by sides by $\ln 3$ to solve for x.
$x = \dfrac{\ln 7}{\ln 3}$.	Note, you answer could also have been $x = \dfrac{\log 7}{\log 3}$, which is the same answer.

EXAMPLE: Solve: $e^{x+5} = 4$.

You definitely want to take the natural log of both sides so we can cancel out the e.

$\ln e^{x+5} = \ln 4$	This is the same as $\log_e e^{x+5} = \ln 4$. We can use property #4 to simplify this:
$x + 5 = \ln 4$	The ln and e cancel, so now we can solve for x.
$x = \ln 4 - 5$	This is our final answer.

If you are wondering if we can do subtract the 5 from 4 then the answer is no. These are not like terms.

Solving Logarithmic Equations

EXAMPLE: Solve: $\log_5(4x+5) = 2$.

To solve this one, we first want to change from logarithmic form to exponential form. You will get $5^2 = 4x + 5$. So we have $25 = 4x + 5$, in which $20 = 4x$, so $x = 5$.

EXAMPLE: Solve: $\log_2(x+11)+\log_2(x+7)=5$

$\log_2(x+11)(x+7)=5$	First combine into one log.
$2^5=(x+11)(x+7)$	Change into exponential form
$32=x^2+18x+77$	Multiply and simplify
$0=x^2+18x+45$	Set it equal to zero
$0=(x+3)(x+15)$	Factor
$x=-3,\ \ x=-15$	These are our answers. Now we need to make sure they are in domain.

If we put -3 into the original we get $\log_2(-3+11)+\log_2(-3+7)=5$ which is $\log_2 8+\log_2 4=5$ This is okay since both 8 and 4 are in the domain. If we put in -15 we get $\log_2(-15+11)+\log_2(-15+7)=5$ which results in $\log_2(-4)+\log_2(-8)=5$. We can't have negative numbers inside a log, therefore -15 is not one of our answers. Our only answer for this problem is $x=-3$.

EXAMPLE: Solve: $\log_2(x+3)-\log_2(x+5)=1$

$\log_2(x+3)-\log_2(x+5)=1$	First combine into one log. This time we will turn it into a fraction
$\log_2\left(\dfrac{x+3}{x+5}\right)=1$	We used property #7. Now change into exponential form.
$\dfrac{x+3}{x+5}=2^1$	We can solve this by cross multiplying.
$2(x+5)=x+3$	Now solve for x.
$2x+10=x+3$	
$x=-7$	

If we put -7 into the original we get $\log_2(-7+3)-\log_2(-7+5)=1$ which is $\log_2(-4)+\log_2(-2)=1$ We can't have a negative inside the log, so we reject the answer x = -7. Since our only answer did not work, the answer to the problem is “no solution” or undefined.