

2.1 The Limit of a Function and Limit Laws

Let's look at the graph $y = \frac{x-2}{x^2-4}$. What is $y(2)$? That's right, it's undefined, but what if we wanted to find the y value the graph is approaching as we get close to an x value of 2? This y -value that it is approaching is called a **limit**. Here's some notation for the problem we were just describing.

$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$ What this means is that we want to find what **y-value** the graph is approaching as x gets close to 2.

Since we need to get really close to an x -value of 2, let's make a table of values. We want to pick values for x that are very close to 2. We can pick a couple above 2 and below 2. To find these, you need to put each x value into the formula $y = \frac{x-2}{x^2-4}$. You can use your table feature on your graphing calculator to do this one.

Because there are many types of graphing calculators, see me after class if you need to know how to use the table function. The table of values should look like:

x	1.9	1.99	1.999	2.001	2.01
y(x)	.2564	.2506	.25001	.2499	.2493

By looking at this table, it appears the y -value is approaching 0.25, or $1/4$. So you would write your answer as:

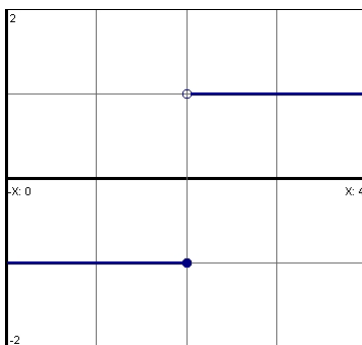
$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$

Now let's use some algebra (more on this later in this section). We can factor the denominator:

$$\frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2} \quad \text{so now the problem becomes } \lim_{x \rightarrow 2} \frac{1}{x+2}. \quad \text{To solve this, just put a 2 in for } x$$

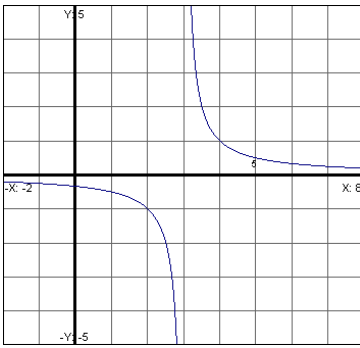
and we get: $\lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$. This is the same as our estimate.

Limits must approach the same number. For example, let's look at the following graph of $f(x)$:



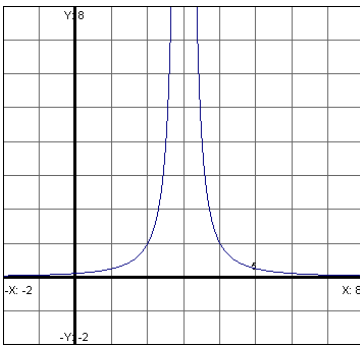
Let's look at $\lim_{x \rightarrow 2} f(x)$. Notice as we approach 2 from the left and from the right we approach two different numbers. When this happens, we say the limit does not exist.

What about $\lim_{x \rightarrow 3} f(x)$ if the graph below is of $f(x)$?



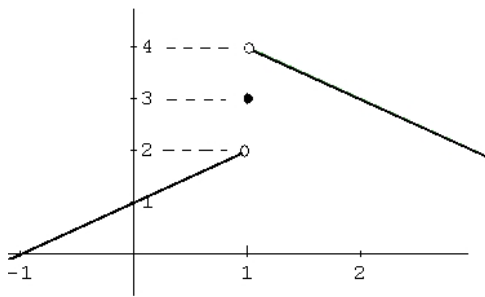
Here as x approaches 3 from each side, the y values approach two different values, so again the limit does not exist.

What about $\lim_{x \rightarrow 3} f(x)$ if the graph below is of $f(x)$?



This one is called an “unbounded limit” since the y -values keep increasing to ∞ . Since infinity is not a real number, the limit does not exist.

Let's look at the graph below:



First, does $f(1)$ exist? Yes, $f(1) = 3$.

What is $\lim_{x \rightarrow 1} f(x)$? Does not exist. Approaches two different y -values.

What is $\lim_{x \rightarrow -1} f(x)$? This exists. As x approaches negative one we want to see what y -value the graph approaches. This would be zero.

More on next page...

Limit Laws

1.) *Sum/Difference Rule:* $\lim_{x \rightarrow c} (f(x) \pm g(x)) = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x)$

2.) *Constant Multiple Rule:* $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot \lim_{x \rightarrow c} f(x)$

3.) *Product Rule:* $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$

4.) *Quotient Rule:* $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$

5.) *Power Rule:* $\lim_{x \rightarrow c} [f(x)]^n = \left[\lim_{x \rightarrow c} f(x) \right]^n$

6.) *Root Rule:* $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)}$

EXAMPLE: Find: $\lim_{x \rightarrow 5} 2x^2 + 7$ by using limit properties to break it down.

The properties tell us we can break up this limit like this:

$$\lim_{x \rightarrow 5} 2x^2 + 7 = \lim_{x \rightarrow 5} 2x^2 + \lim_{x \rightarrow 5} 7 = 2 \lim_{x \rightarrow 5} x^2 + \lim_{x \rightarrow 5} 7 = 2(5)^2 + 7 = 57$$

What are we really doing here? We are simply plugging in the c value into our expression. You are not required to show the break down of each limit unless the questions specifically ask you to.

EXAMPLE: Find: $\lim_{x \rightarrow 2} \frac{3x^2 - x + 2}{x + 2}$.

Here, we are just going to replace x with 2 since this is our c value.

$$\lim_{x \rightarrow 2} \frac{3x^2 - x + 2}{x + 2} = \frac{3(2)^2 - (2) + 2}{(2) + 2} = \frac{12}{4} = 3$$

EXAMPLE: Find: $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 - 9}$.

The problem with this one is that if I put in a -3 for x I will be dividing by zero which is undefined. However if we factor the denominator you will be able to cancel out the part that make the bottom zero. Once this part is eliminated then we can plug in the -3 for x:

$$\lim_{x \rightarrow -3} \frac{x + 3}{x^2 - 9} = \lim_{x \rightarrow -3} \frac{x + 3}{(x + 3)(x - 3)} = \lim_{x \rightarrow -3} \frac{1}{x - 3} = \frac{1}{-3 - 3} = -\frac{1}{6}$$

EXAMPLE: Find: $\lim_{x \rightarrow 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8}$.

This is another one where you will divide by zero if you put in a 4 for x. Again you would want to factor both the numerator and denominator and then cancel. Finally you can then plug in 4 for x.

$$\lim_{x \rightarrow 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x \rightarrow 4} \frac{(x-1)(x-4)}{(x+2)(x-4)} = \lim_{x \rightarrow 4} \frac{x-1}{x+2} = \frac{4-1}{4+2} = \frac{3}{6} = \frac{1}{2}$$

EXAMPLE: Find: $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}}{x-4}$.

Since plugging in a 3 won't give us a zero in the denominator, we can just plug in 3 and get the answer:

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}}{x-4} = \frac{\sqrt{3+1}}{3-4} = \frac{\sqrt{4}}{-1} = -2$$

EXAMPLE: Find $\lim_{x \rightarrow -31} \sqrt[3]{x+4}$

For this one, just plug in -31 for x. Remember you are allowed to take the odd root of a negative number.

$$\lim_{x \rightarrow -31} \sqrt[3]{x+4} = \sqrt[3]{-31+4} = \sqrt[3]{-27} = -3$$

EXAMPLE: Find: $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

The problem with this one is that if we put in a zero for x we will be dividing by zero so we must do something to this to get rid of the x. Almost always the operation you will do is to multiply the top and bottom by the conjugate. A conjugate (of the numerator in this case) is the same thing but with the opposite sign. So we will multiply top and bottom by $\sqrt{2+x} + \sqrt{2}$. Then we will cancel:

$$\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}}$$

Now when we multiply across the top you can use the difference of squares formula, which is $(a-b)(a+b) = a^2 - b^2$. So if we have $(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})$ then this will equal:

$(\sqrt{2+x})^2 - (\sqrt{2})^2$ and when we simplify we get $(2+x) - (2) = x$. So let's continue:

$$\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{2+x} + \sqrt{2})}$$
 Now cancel the x's to get:

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{2+x} + \sqrt{2}}. \text{ Now plug in 0 for x: } \lim_{x \rightarrow 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{\sqrt{2+0} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2\sqrt{4}} = \frac{\sqrt{2}}{4}$$

EXAMPLE: Find: $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2}$.

This is another one we need to multiply by the conjugate, but this time we will multiply by the conjugate of the denominator, which would be $\sqrt{x+1}+2$. We then will follow the same steps as shown above. You can still use the difference of squares formula when you do $(\sqrt{x+1}-2)(\sqrt{x+1}+2)$.

$$\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2} = \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3}$$

$$\lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} = \lim_{x \rightarrow 3} \sqrt{x+1}+2 = \sqrt{3+1}+2 = 4$$

Limits with Trigonometric Functions

With these limits we can still plug in the c value into the expression to get the limit. You will just need to make sure you have your unit circle or trig tables ready.

EXAMPLE: Find $\lim_{x \rightarrow \pi} \tan x$

Just put in the pi for x. You will get: $\lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$

EXAMPLE: Find $\lim_{x \rightarrow \frac{\pi}{6}} \cos 3x$

$$\lim_{x \rightarrow \frac{\pi}{6}} \cos 3x = \cos \left(3 \cdot \frac{\pi}{6} \right) = \cos \left(\frac{\pi}{2} \right) = 0$$

EXAMPLE: Find $\lim_{x \rightarrow \frac{\pi}{3}} 3 \sin x$

$$\lim_{x \rightarrow \frac{\pi}{3}} 3 \sin x = 3 \sin \left(\frac{\pi}{3} \right) = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Sometimes you may need to use trig identities to simplify before you plug in.

EXAMPLE: Find $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\tan \theta}{\sec \theta}$

If we plug in pi right now we will be dividing by zero. We will write these in terms of sine and cosine.

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\tan \theta}{\sec \theta} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos \theta}} = \lim_{\theta \rightarrow \frac{\pi}{2}} \sin \theta = \sin \left(\frac{\pi}{2} \right) = 1.$$

Sandwich (Squeeze) Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then $\lim_{x \rightarrow c} f(x) = L$.

EXAMPLE: Find $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right)$.

Here we see that if we plug in zero then the function is undefined. So we are going to apply the Sandwich Theorem here. We want to find functions that are similar to enough to $x^2 \cos\left(\frac{1}{x^2}\right)$ but will be easy to evaluate as x approaches 0. Let's start with the most complicated part of $x^2 \cos\left(\frac{1}{x^2}\right)$. In this case it is $\cos\left(\frac{1}{x^2}\right)$. We know the cosine stays between -1 and 1, so $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$ for any x except when $x = 0$. Since x^2 is always positive, we can multiply this inequality through by x^2 . We will get $-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$. So our original function is bounded by $-x^2$ and x^2 . Now since $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$, then, by the Sandwich Theorem,

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0.$$

EXAMPLE: If $3x - 1 \leq f(x) \leq x^2 + 1$, find $\lim_{x \rightarrow 2} f(x)$ using the Squeeze Theorem.

For this problem, we need to find the limit of each of the given functions as x approaches 2:

$$\lim_{x \rightarrow 2} 3x - 1 = 3(2) - 1 = 5$$

$$\lim_{x \rightarrow 2} x^2 + 1 = 2^2 + 1 = 5$$

Since $\lim_{x \rightarrow 2} 3x - 1 = \lim_{x \rightarrow 2} x^2 + 1 = 5$, then $\lim_{x \rightarrow 2} f(x) = 5$ by the Squeeze Theorem.