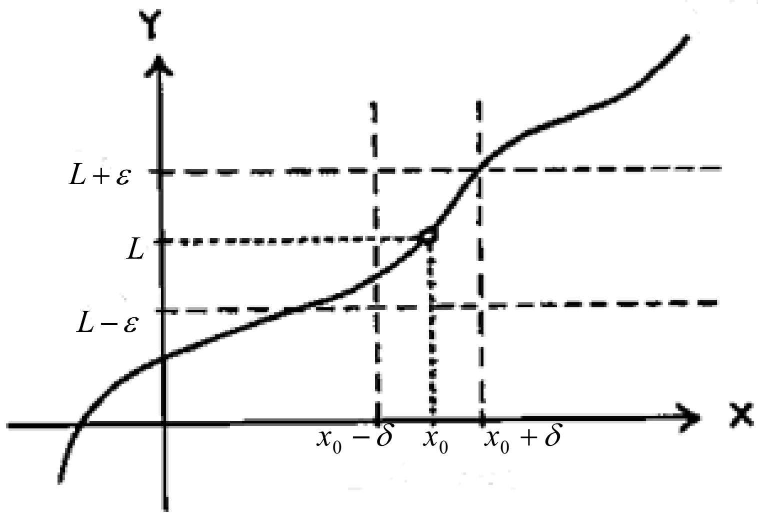
2.2 The Precise Definition of a Limit

Precise definition of a limit

Now we will look at the definition of a limit. Suppose we want to find $\lim_{x \to x_0} f(x)$. We are going to let δ represent an increment. This will tell us how close we are to x_0. We might be, say, 0.001 off from x_0 with some measuring error. We will let L represent the actual value of the limit. Now depending how much we are off from our measurement of c this will affect the values that we get for the limit. We might be off on this measurement as well. Let's call ε the amount we are off from the limit. So, δ corresponds to x values and ε represents y values. See the picture on the next page. This shows the relationship between all these variables we just mentioned.

What this is saying below is how close to x_0 does x have to be so that f(x) differs from L by less than ε units?



Now that we have defined these variables, let's look at the definition of the limit.

Let f be defined on an open interval containing c and let L be a real number. Then:

$\lim_{x \to x_0} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.
What this means is that if we say an error in how far we are from the actual limit ε, then there is also an error coming from the x-direction, which is δ. The positive distance between the x value and x_0 has to be larger than zero but less than δ. If this is true then the difference between the actual y-value of the limit and the y-value of the error f(x) must be greater than zero, but less than ε..

EXAMPLE: Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ such that a < x < b. Given: a = 1, b = 7, $x_0 = 2$.

You would sketch this interval on a number line, you will get: (1__2________7). The value of δ is the difference between x_0 and the CLOSEST endpoint. If we take $x_0 - a$, we get 1. If we take $b - x_0$, we get 5. Therefore δ would have to be the smaller of these two results, so $\delta = 1$.

EXAMPLE: Use the ε-δ definition of a limit to prove that $\lim_{x \to 3} 5x - 4 = 11$.

Before we write the proof, let's define our variables. $x_0 = 3$, L = 11, and f(x) = 5x – 4. The following rest of what follows is the answer. This answer is actually a series of statements that make up a proof. Here we go:

<u>Proof:</u>

For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|5x-4-11| < \varepsilon$.

$|(5x-4)-11| < \varepsilon$

$|5(x-3)| < \varepsilon$

$5|x-3| < \varepsilon$ so $|x-3| < \frac{\varepsilon}{5}$. Hence, let $\delta = \frac{\varepsilon}{5}$.

Hence, if $0 < |x-3| < \delta = \frac{\varepsilon}{5}$

$|x-3| < \frac{\varepsilon}{5}$, or

$5|x-3| < \varepsilon$, or

$|5x-15| < \varepsilon$, or

$|(5x-4)-11| < \varepsilon$, or

$|f(x)-L| < \varepsilon$

EXAMPLE: Use the ε-δ definition of a limit to prove that $\lim_{x \to 4} \frac{x}{2} + 6 = 8$.

Before we write the proof, let's define our variables. $x_0 = 4$, L = 8, and $f(x) = \frac{x}{2} + 6$. The following rest of what follows is the answer. This answer is actually a series of statements that make up a proof. Here we go:

<u>Proof:</u>

For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-4| < \delta$, then $\left|\frac{x}{2}+6-8\right| < \varepsilon$.

$\left|\frac{x}{2}+6-8\right| < \varepsilon$

$\left|\frac{x}{2}-2\right| < \varepsilon$

$\left|\frac{1}{2}(x-4)\right| < \varepsilon$

$\frac{1}{2}|x-4| < \varepsilon$

$|x-4| < 2\varepsilon$

Hence, let $\delta = 2\varepsilon$.

Hence, if $0 < |x-4| < \delta = 2\varepsilon$

$|x-4| < 2\varepsilon$

$\frac{1}{2}|x-4| < \varepsilon$

$\left|\frac{1}{2}(x-4)\right| < \varepsilon$, so $\left|\frac{x}{2}-2\right| < \varepsilon$ and $\left|\frac{x}{2}+6-8\right| < \varepsilon$ and

$|f(x)-L| < \varepsilon$

EXAMPLE: In the following exercises, find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give the largest value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds:

a.) $f(x)=2x-2$, L = -6, $x_0=-2$, $\varepsilon = 0.02$

$|2x-2-(-6)|<0.02$ — First we plug in our values into $|f(x)-L|<\varepsilon$. Then we solve for x.

$|2x+4|<0.02$

$-0.02<2x+4<0.02$

$-4.02<2x<-3.98$

$-2.01<x<-1.99$

$(-2.01, \ -1.99)$ — Here is our open interval answer.

For the second part of the question, let's first start with $0<|x-x_0|<\delta$. We are given that $x_0=-2$, so this becomes $0<|x+2|<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $|x+2|<\delta$. Now let's solve this for x: $-\delta<x+2<\delta$. Now subtract 2 from both sides to get: $-2-\delta<x<\delta-2$. From above, we solved for x and got: $-2.01<x<-1.99$. Therefore we know that $-2-\delta=-2.01$ or $\delta-2=-1.99$. Solving either one of these will get $\delta=0.01$. Instead of doing all of this, we could have just noticed that the interval $-2.01<x<-1.99$ is centered around our x_0, which was 2. We notice that the distance from either endpoint to the number 2 is 0.01, so we know $\delta=0.01$

b.) $f(x)=\sqrt{x-7}$, L = 4, $x_0=23$, $\varepsilon=1$

$\left|\sqrt{x-7}-4\right|<1$ — First we plug in our values into $|f(x)-L|<\varepsilon$. Then we solve for x.

$-1<\sqrt{x-7}-4<1$

$3<\sqrt{x-7}<5$ — Now square both sides.

$9<x-7<25$

$16<x<32$

$(16, \ 32)$ — Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0=23$, so this becomes $0<|x-23|<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $|x-23|<\delta$. Now let's solve this for x: $-\delta<x-23<\delta$. Now add 23 to both sides to get: $23-\delta<x<\delta+23$. So we have $23-\delta=16$ and $\delta+23=32$. Solving both of these we get $\delta=7$ and $\delta=9$. You want to minimize the error in the x direction, so we want the smaller answer. Therefore, our answer is going to be $\delta=7$.

c.) $f(x)=1/x$, L = -1, $x_0=-1$, $\varepsilon=0.1$

$\lvert 1/x+1\rvert<0.1$	First we plug in our values into $\lvert f(x)-L\rvert<\varepsilon$. Then we solve for x.
$-0.1<1/x+1<0.1$	Let's write this in terms of fractions.
$-\frac{1}{10}<1/x+1<\frac{1}{10}$	Now subtract 1 from all sides.
$-\frac{11}{10}<1/x<-\frac{9}{10}$	Next we need to take the reciprocal of both sides and reverse the inequality.
$-\frac{10}{11}>x>-\frac{10}{9}$	
$\left(-\frac{10}{9},\ -\frac{10}{11}\right)$	Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0=-1$, so this becomes $0<\lvert x+1\rvert<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $\lvert x+1\rvert<\delta$. Now let's solve this for x: $-\delta<x+1<\delta$. Now subtract 1 from both sides to get: $-\delta-1<x<\delta-1$. So we have $-\delta-1=-\frac{10}{9}$ and $\delta-1=-\frac{10}{11}$. Solving both of these we get $\delta=\frac{1}{9}$ and $\delta=\frac{1}{11}$. You want to minimize the error in the x direction, so we want the smaller answer. Therefore, our answer is going to be $\delta=\frac{1}{11}$.

d.) $f(x)=x^2$, L = 3, $x_0=\sqrt{3}$, $\varepsilon=0.1$ (Round answers to four decimal places)

$\lvert x^2-3\rvert<0.1$	First we plug in our values into $\lvert f(x)-L\rvert<\varepsilon$. Then we solve for x.
$-0.1<x^2-3<0.1$	We add 3 to both sides of the equation.
$2.9<x^2<3.1$	Now square root both sides.
$1.7029<x<1.7607$	
$(1.7029,\ 1.7607)$	Here is our open interval answer.

We are given that $x_0=\sqrt{3}$, so this becomes $0<\lvert x-\sqrt{3}\rvert<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $\lvert x-\sqrt{3}\rvert<\delta$. Now let's solve this for x: $-\delta<x-\sqrt{3}<\delta$. Now add $\sqrt{3}$ to both sides to get: $\sqrt{3}-\delta<x<\delta+\sqrt{3}$. As decimals this can be written as: $1.7321-\delta<x<\delta+1.7321$. So we have $1.7321-\delta=1.7029$ and $\delta+1.7321=1.7607$. Solving these we get $\delta=0.0292$ and $\delta=0.0286$, so we want the smaller answer. Therefore $\delta=0.0286$.