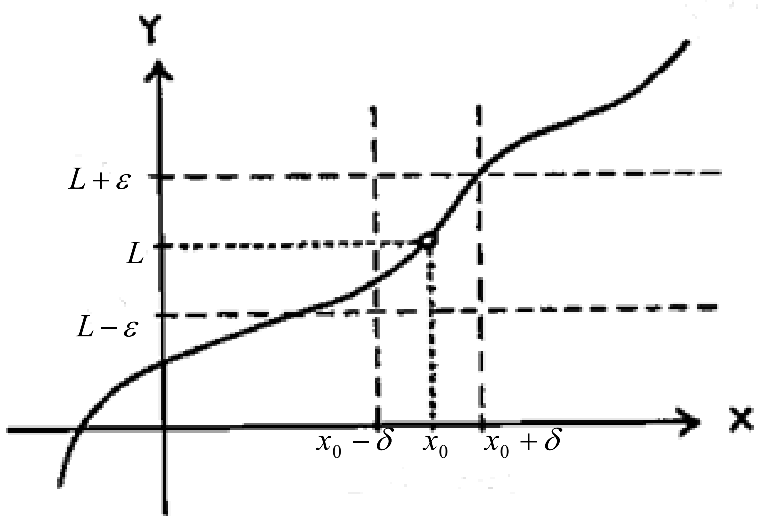
2.2 The Precise Definition of a Limit

Precise definition of a limit

Now we will look at the definition of a limit. Suppose we want to find $\lim_{x \to x_0} f(x)$. We are going to let δ represent an increment. This will tell us how close we are to x_0. We might be, say, 0.001 off from x_0 with some measuring error. We will let L represent the actual value of the limit. Now depending how much we are off from our measurement of c this will affect the values that we get for the limit. We might be off on this measurement as well. Let's call ε the amount we are off from the limit. So, δ corresponds to x values and ε represents y values. See the picture on the next page. This shows the relationship between all these variables we just mentioned.

What this is saying below is how close to x_0 does x have to be so that f(x) differs from L by less than ε units?



Now that we have defined these variables, let's look at the definition of the limit.

Let f be defined on an open interval containing c and let L be a real number. Then:

$\lim_{x \to x_0} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

What this means is that if we say an error in how far we are from the actual limit ε, then there is also an error coming from the x-direction, which is δ. The positive distance between the x value and x_0 has to be larger than zero but less than δ. If this is true then the difference between the actual y-value of the limit and the y-value of the error f(x) must be greater than zero, but less than ε..

EXAMPLE: Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ such that a < x < b. Given: a = 1, b = 7, $x_0 = 2$.

You would sketch this interval on a number line, you will get: (1__2________7). The value of δ is the difference between x_0 and the CLOSEST endpoint. If we take $x_0 - a$, we get 1. If we take $b - x_0$, we get 5. Therefore δ would have to be the smaller of these two results, so $\delta = 1$.

EXAMPLE: Use the ε - δ definition of a limit to prove that $\lim_{x\to 3} 5x-4=11$.

Before we write the proof, let's define our variables. $x_0=3$, L = 11, and f(x) = 5x – 4. We also want to do our scratchwork first so that we know what our δ needs to be:

Proof:

For each $\varepsilon>0$ there exists a $\delta>0$ such that if

$0<|x-3|<\delta$, then $|5x-4-11|<\varepsilon$.

Let $\delta=\frac{\varepsilon}{5}$. Then

$0<|x-3|<\delta=\frac{\varepsilon}{5}$

$|x-3|<\frac{\varepsilon}{5}$

$5|x-3|<\varepsilon$

$|5x-15|<\varepsilon$

$|(5x-4)-11|<\varepsilon$

$|f(x)-L|<\varepsilon$

Scratchwork:

$|(5x-4)-11|<\varepsilon$

$|5(x-3)|<\varepsilon$

$5|x-3|<\varepsilon$

$|x-3|<\frac{\varepsilon}{5}$

EXAMPLE: Use the ε - δ definition of a limit to prove that $\lim_{x\to -4} \frac{x}{2}+6=4$.

Here: $x_0=-4$, L = 4, and $f(x)=\frac{x}{2}+6$.

Proof:

For each $\varepsilon>0$ there exists a $\delta>0$ such that if

$0<|x-(-4)|<\delta$, then $\left|\frac{x}{2}+6-4\right|<\varepsilon$.

Let $\delta=2\varepsilon$. Then,

$0<|x+4|<\delta=2\varepsilon$

$|x+4|<2\varepsilon$

$\frac{1}{2}|x+4|<\varepsilon$

$\left|\frac{1}{2}(x+4)\right|<\varepsilon$

$\left|\frac{x}{2}+2\right|<\varepsilon$

$\left|\frac{x}{2}+6-4\right|<\varepsilon$,

$|f(x)-L|<\varepsilon$

Scratchwork:

$\left|\frac{x}{2}+6-4\right|<\varepsilon$

$\left|\frac{x}{2}+2\right|<\varepsilon$

$\left|\frac{1}{2}(x+4)\right|<\varepsilon$

$\frac{1}{2}|x+4|<\varepsilon$

$|x+4|<2\varepsilon$

EXAMPLE: In the following exercises, find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give the largest value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds:

a.) $f(x)=2x-2$, L = -6, $x_0=-2$, $\varepsilon=0.02$

$\lvert 2x-2-(-6)\rvert<0.02$	First we plug in our values into $\lvert f(x)-L\rvert<\varepsilon$. Then we solve for x.
$\lvert 2x+4\rvert<0.02$	
$-0.02<2x+4<0.02$	
$-4.02<2x<-3.98$	
$-2.01<x<-1.99$	
$(-2.01,\ -1.99)$	Here is our open interval answer.

For the second part of the question, let's first start with $0<|x-x_0|<\delta$. We are given that $x_0=-2$, so this becomes $0<|x+2|<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $|x+2|<\delta$. Now let's solve this for x: $-\delta<x+2<\delta$. Now subtract 2 from both sides to get: $-2-\delta<x<\delta-2$. From above, we solved for x and got: $-2.01<x<-1.99$. Therefore we know that $-2-\delta=-2.01$ or $\delta-2=-1.99$. Solving either one of these will get $\delta=0.01$. Instead of doing all of this, we could have just noticed that the interval $-2.01<x<-1.99$ is centered around our x_0, which was 2. We notice that the distance from either endpoint to the number 2 is 0.01, so we know $\delta=0.01$

b.) $f(x)=\sqrt{x-7}$, L = 4, $x_0=23$, $\varepsilon=1$

$\lvert\sqrt{x-7}-4\rvert<1$	First we plug in our values into $\lvert f(x)-L\rvert<\varepsilon$. Then we solve for x.
$-1<\sqrt{x-7}-4<1$	
$3<\sqrt{x-7}<5$	Now square both sides.
$9<x-7<25$	
$16<x<32$	
$(16,\ 32)$	Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0=23$, so this becomes $0<|x-23|<\delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $|x-23|<\delta$. Now let's solve this for x: $-\delta<x-23<\delta$. Now add 23 to both sides to get: $23-\delta<x<\delta+23$. So we have $23-\delta=16$ and $\delta+23=32$. Solving both of these we get $\delta=7$ and $\delta=9$. You want to minimize the error in the x direction, so we want the smaller answer. Therefore, our answer is going to be $\delta=7$.

c.) $f(x) = 1/x$, L = -1, $x_0 = -1$, $\varepsilon = 0.1$

$\lvert 1/x+1\rvert < 0.1$	First we plug in our values into $\lvert f(x) - L\rvert < \varepsilon$. Then we solve for x.
$-0.1 < 1/x+1 < 0.1$	Let's write this in terms of fractions.
$-\frac{1}{10} < 1/x+1 < \frac{1}{10}$	Now subtract 1 from all sides.
$-\frac{11}{10} < 1/x < -\frac{9}{10}$	Next we need to take the reciprocal of both sides and reverse the inequality.
$-\frac{10}{11} > x > -\frac{10}{9}$	
$\left(-\frac{10}{9}, -\frac{10}{11}\right)$	Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0 = -1$, so this becomes $0 < \lvert x+1\rvert < \delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $\lvert x+1\rvert < \delta$. Now let's solve this for x: $-\delta < x+1 < \delta$. Now subtract 1 from both sides to get: $-\delta - 1 < x < \delta - 1$. So we have $-\delta - 1 = -\frac{10}{9}$ and $\delta - 1 = -\frac{10}{11}$. Solving both of these we get $\delta = \frac{1}{9}$ and $\delta = \frac{1}{11}$. You want to minimize the error in the x direction, so we want the smaller answer. Therefore, our answer is going to be $\delta = \frac{1}{11}$.

d.) $f(x) = x^2$, L = 3, $x_0 = \sqrt{3}$, $\varepsilon = 0.1$ (Round answers to four decimal places)

$\lvert x^2 - 3\rvert < 0.1$	First we plug in our values into $\lvert f(x) - L\rvert < \varepsilon$. Then we solve for x.
$-0.1 < x^2 - 3 < 0.1$	We add 3 to both sides of the equation.
$2.9 < x^2 < 3.1$	Now square root both sides.
$1.7029 < x < 1.7607$	
$(1.7029,\ 1.7607)$	Here is our open interval answer.

We are given that $x_0 = \sqrt{3}$, so this becomes $0 < \lvert x - \sqrt{3}\rvert < \delta$ Now let's solve this for x. We know that absolute values are greater than 0, so this problem can be simplified to: $\lvert x - \sqrt{3}\rvert < \delta$. Now let's solve this for x: $-\delta < x - \sqrt{3} < \delta$. Now add $\sqrt{3}$ to both sides to get: $\sqrt{3} - \delta < x < \delta + \sqrt{3}$. As decimals this can be written as: $1.7321 - \delta < x < \delta + 1.7321$. So we have $1.7321 - \delta = 1.7029$ and $\delta + 1.7321 = 1.7607$. Solving these we get $\delta = 0.0292$ and $\delta = 0.0286$, so we want the smaller answer. Therefore $\delta = 0.0286$.