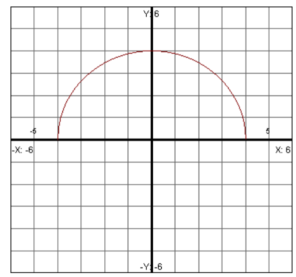
2.3 One-Sided Limits

One-Sided Limits

$\lim\limits_{x \to c^+} f(x) = L$ This means we are finding the limit of f as we approach c from the right (positive side)

$\lim\limits_{x \to c^-} f(x) = L$ This means we are finding the limit of f as we approach c from the left (negative side)

EXAMPLE: Find the limit: $\lim\limits_{x \to 4^-} \sqrt{16 - x^2}$



We can still put in a 4 for x to get: $\sqrt{16 - 4^2} = 0$
Look at the graph to the right. As we approach 4 from the right, The y-value is approaching 0.

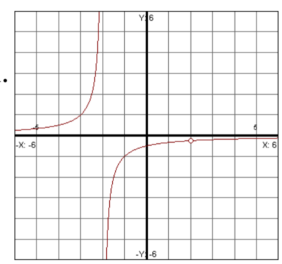
EXAMPLE: Find the limit: $\lim\limits_{x \to 2^+} \dfrac{2 - x}{x^2 - 4}$

$\lim\limits_{x \to 2^+} \dfrac{2 - x}{(x-2)(x+2)}$ First we factor the denominator. We can't quite cancel yet. We will factor once more.

$\lim\limits_{x \to 2^+} \dfrac{-(-2 + x)}{(x-2)(x+2)}$ I factored out a negative one from the numerator. Now I can cancel with x – 2.

$\lim\limits_{x \to 2^+} \dfrac{-1}{(x+2)}$ We still need to keep the negative on top. Now plug in 2 for x.

$\dfrac{-1}{2+2} = -\dfrac{1}{4}$. Now see the graph to verify our answer:



Finding One-Sided Limits Algebraically

EXAMPLE: Find the limit: $\lim\limits_{x \to 1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$.

Even though this is a one-sided limit, we can just plug in the value x = 1 for x into our expression just like we did before:

$$\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right) = \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = 1$$

So we know that $\lim\limits_{x \to 1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right) = 1$.

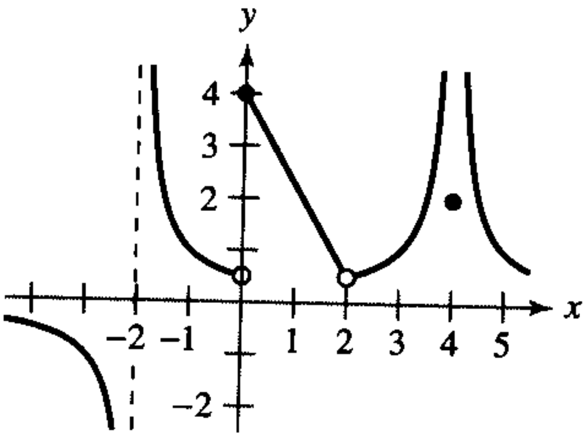
Now what happens if we plug in our value and we get a zero in the denominator? Well this won't happen in this section, however we will look at this in the next section. There are special ways of handling these types of problems.

Now let's read values off a graph using our new notation for one-sided limits:

EXAMPLE: Use the graph of f(x) below to find the following:

$$f(0),\ f(2),\ f(-2),\ f(4),\ \lim_{x\to 2^+} f(x),\ \lim_{x\to 2^-} f(x),\ \lim_{x\to 2} f(x),\ \lim_{x\to 0^-} f(x),\ \lim_{x\to 0^+} f(x),\ \lim_{x\to 0} f(x),$$

$$\lim_{x\to -2} f(x)$$



a.) $f(0) = 4$ Notice you are finding the y-value when x = 0. A closed circle is where the graph is defined.

b.) $f(2) = \text{undef.}$ For this one, there is no closed circle at the x = 2. So, nothing is defined here.

c.) $f(-2) = \text{undef.}$ There is no closed circles here either. We have a vertical asymptote, so nothing will be defined here.

d.) $f(4) = 2$ You are finding the y-value when x = 4.

e.) $\lim_{x\to 2^+} f(x) = \frac{1}{2}$ You are seeing what the y-value is approaching as x approaches 2 from the right.

f.) $\lim_{x\to 2^-} f(x) = \frac{1}{2}$ You are seeing what the y-value is approaching as x approaches 2 from the left.

g.) $\lim_{x\to 2} f(x) = \frac{1}{2}$ Since the limit from the left and right are the same then our limit exists and is also equal to 1.

h.) $\lim_{x\to 0^+} f(x) = 4$ You are seeing what the y-value is approaching as x approaches 0 from the right.

i.) $\lim_{x\to 0^-} f(x) = \frac{1}{2}$ You are seeing what the y-value is approaching as x approaches 0 from the left.

j.) $\lim_{x\to 0} f(x) =$ d.n.e. Since the limit from the left and from the right are not the same, the limit does not exist.

k.) $\lim_{x\to -2} f(x) =$ d.n.e. Since the limit from the left and from the right are not the same, the limit does not exist.

Special Trigonometric Limits

If n and k are real numbers and θ is in radians, then:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \qquad \lim_{\theta\to 0}\frac{\sin\left(\theta^n\right)}{\theta^n}=1 \qquad \lim_{\theta\to 0}\frac{\sin\left(k\theta\right)}{k\theta}=1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0 \qquad \lim_{\theta\to 0}\frac{1-\cos\left(\theta^n\right)}{\theta^n}=0 \qquad \lim_{\theta\to 0}\frac{1-\cos\left(k\theta\right)}{k\theta}=0$$

If you were to look at the graph of this function on your calculator, you will see that the left and right handed limit approaches 1 where θ is in radians.

EXAMPLE: Find $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{23\theta}$.

The problem with this one is that we can't just put down one for the answer because the top and bottom do not match. We need the bottom to be θ. Here, we can write the problem this way:

$$\lim_{\theta\to 0}\frac{1}{23}\cdot\frac{\sin\theta}{\theta}$$

Now we can move the fraction 1/23 out in front of the limit:

$$\frac{1}{23}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

From here, apply the limit $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

$$\frac{1}{23}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{1}{23}(1)=\frac{1}{23}$$

EXAMPLE: Find $\lim\limits_{\theta\to 0}\dfrac{\sin 8\theta}{5\theta}$.

We have another problem in which the expression inside the sine does not match the bottom. We need the bottom to be 8θ. We can do this by multiplying the top and bottom by $1/8\theta$:

$$\lim_{\theta\to 0}\frac{\sin 8\theta}{5\theta}=\lim_{\theta\to 0}\frac{\sin 8\theta}{5\theta}\cdot\frac{\left(\dfrac{1}{8\theta}\right)}{\left(\dfrac{1}{8\theta}\right)}=\lim_{\theta\to 0}\frac{\dfrac{\sin 8\theta}{8\theta}}{\dfrac{5}{8}}=\frac{8}{5}\cdot\lim_{\theta\to 0}\frac{\sin 8\theta}{8\theta}=\frac{8}{5}(1)=\frac{8}{5}$$

The 5/8 on the bottom can be put in front of the limit by using a limit property. Then $\lim\limits_{\theta\to 0}\dfrac{\sin 8\theta}{8\theta}=1$

EXAMPLE: Find $\lim\limits_{\theta \to 0} \dfrac{1-\cos 3\theta}{6\theta}$

We will use the same technique as in the previous example. We want to get 3θ in the top and bottom so we need to divide the top and bottom by 2 so we can get a 3θ in the denominator.

$$\lim_{\theta \to 0} \frac{1-\cos 3\theta}{6\theta} = \lim_{\theta \to 0} \frac{\frac{1-\cos 3\theta}{2}}{\frac{6\theta}{2}} = \lim_{\theta \to 0} \frac{\frac{1}{2}\cdot\left(1-\cos 3\theta\right)}{3\theta} = \frac{1}{2}\cdot \lim_{\theta \to 0} \frac{1-\cos 3\theta}{3\theta} = \frac{1}{2}\cdot 0 = 0$$

EXAMPLE: Find $\lim\limits_{\theta \to 0} \dfrac{1-\cos(2\theta)+\sin(3\theta)}{2\theta}$.

Here, we will divide the first two terms by 2θ and then the last term by 2θ. we will get:

$$\lim_{\theta \to 0} \frac{1-\cos(2\theta)}{2\theta} + \frac{\sin(3\theta)}{2\theta}.$$

We can apply the limit to each term separately because of the limit rules:

$$\lim_{\theta \to 0} \frac{1-\cos(2\theta)}{2\theta} + \lim_{\theta \to 0} \frac{\sin(3\theta)}{2\theta}$$

$$\lim_{\theta \to 0} \frac{1-\cos(2\theta)}{2\theta} + \lim_{\theta \to 0} \frac{\sin(3\theta)}{2\theta}\cdot\frac{\left(\frac{1}{3\theta}\right)}{\left(\frac{1}{3\theta}\right)}$$

We want the top to be $\dfrac{\sin(3\theta)}{3\theta}$, so that is why we multiply.

$$= \lim_{\theta \to 0} \frac{1-\cos(2\theta)}{2\theta} + \lim_{\theta \to 0} \frac{\frac{\sin(3\theta)}{3\theta}}{\frac{2}{3}} = \lim_{\theta \to 0} \frac{1-\cos(2\theta)}{2\theta} + \frac{3}{2}\lim_{\theta \to 0} \frac{\sin(3\theta)}{3\theta}$$

From here, apply special limits.

$$= 0 + \frac{3}{2}(1) = \frac{3}{2}$$

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EXAMPLE: Find $\lim_{x \to 0} \frac{\sin x - \sin x \cos 3x}{\left(3x^2 \sin^2 x + 3x^2 \cos^2 x\right)}$

For this one we first need to factor the numerator and denominator.

$\lim_{\theta \to 0} \frac{\sin x(1 - \cos 3x)}{3x^2(\sin^2 x + \cos^2 x)}$ We know that $\sin^2 x + \cos^2 x = 1$. So now we have:

$\lim_{\theta \to 0} \frac{\sin x(1 - \cos 3x)}{3x^2}$ We will separate this into the product of two different limits using the property.

$\lim_{\theta \to 0} \frac{\sin x}{x} \cdot \lim_{\theta \to 0} \frac{(1 - \cos 3x)}{3x}$. Now evaluate each limit: $\lim_{\theta \to 0} \frac{\sin x}{x} \cdot \lim_{\theta \to 0} \frac{(1 - \cos 3x)}{3x} = 1 \cdot 0 = 0$.