2.5 Limits Involving Infinity; Asymptotes of Graphs

Sometimes limits don't always go to a single number. Sometimes they may go to infinity or negative infinity. You do not need to graph every function. You can utilize tables. Let's look at some examples:

EXAMPLE: Use the table below for $f(x)=\dfrac{5}{x+3}$ and find: $\lim\limits_{x\to -3^+} f(x)$, $\lim\limits_{x\to -3^-} f(x)$, $\lim\limits_{x\to -3} f(x)$

x	-3.5	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9	-2.5	
f(x)	-10	-50	-500	-5000	undef.	5000	500	50	10	

a.) $\lim\limits_{x\to -3^-} f(x)$ This means we need to look at what the f(x) value is approaching as I get close to 3 from the negative side. This means we look x values to the left of -3, or values smaller than -3. This includes all x values that begin with a -3. As I get closer to -3 the value are getting smaller. For example, when x is -3.001 the f(x) value is -5000. From this I can conclude that the values will be approaching negative infinity $-\infty$ which is our answer.

b.) $\lim\limits_{x\to -3^+} f(x)$ This means we need to look at what the f(x) value is approaching as I get close to 3 from the positive side. This means we look x values to the right of -3, or values bigger than -3. This includes all x values that begin with a -2. As I get closer to -3 the value are getting larger. For example, when x is -2.999 the f(x) value is 5000. From this I can conclude that the values will be approaching positive infinity ∞ which is our answer.

c.) $\lim\limits_{x\to -3} f(x)$ From parts a and b we can conclude that this limit does not exist (d.n.e.) since it approaches two different values from the left and from the right.

EXAMPLE: Find $\lim\limits_{x\to 1^+}\dfrac{2+x}{1-x}$

This one you do not need to draw a graph. We also can't cancel anything and we can not plug in 1 one since the bottom will be zero. For this I suggest using a test value for x. As in the example above we want to choose a value that is close to 1. Since we are approaching 1 from the right we a value that is slightly larger than one. You can pick whatever value you want, but I will pick x = 1.0001. If we put this in for x, we get:
$\dfrac{2+1.0001}{1-1.0001}=-30001$. This is a very large negative number. Our conclusion is that the limit is approaching negative infinity $-\infty$.

EXAMPLE: Find $\lim\limits_{x\to 0^-} x^2-\dfrac{1}{x}$

Again we want to use a test value for x. Since we are approaching 0 from the left, we want to pick a value that is slightly less than 0. You can pick whatever value you want, but I will pick x = -0.0001:
$(-0.0001)^2-\dfrac{1}{-0.0001}=10000$. This is a large positive number. Our conclusion is that the limit is approaching positive infinity ∞.

EXAMPLE: Find $\lim\limits_{x\to\frac{\pi}{2}^+}\dfrac{-2}{\cos x}$

This one you do need a test value since we can't cancel anything out and we will get a zero in the denominator if we plug in $\dfrac{\pi}{2}$. It might be easier to first get the decimal of $\dfrac{\pi}{2}$. It is about 1.57 if we check this on a calculator. We need to approach this decimal from the positive side, so we should check a number that is slightly above 1.58. Again you can pick any number, but I will use x = 1.58:
$\dfrac{-2}{\cos(1.58)}\approx 217$. This is a large positive number, so our conclusion is that the limit is approaching $+\infty$.

EXAMPLE: Find $\lim\limits_{x\to 4^-}\dfrac{x^2}{x^2+16}$

Do you need to use a test point for this one? No, because you do not get a zero in the denominator if you plug in 4 for x. Therefore we can just plug in 4 to get the answer:
$\lim\limits_{x\to 4^-}\dfrac{x^2}{x^2+16}=\dfrac{4^2}{(4)^2+16}=\dfrac{16}{32}=\dfrac{1}{2}$. Notice that even though we are approaching 4 from the left, we can still just put 4 into the expression.

EXAMPLE: Find $\lim\limits_{x\to-\frac{1}{2}^+}\dfrac{6x^2+x-1}{4x^2-4x-3}$

What about a test value for this one? Well even though ½ will cause the denominator to be zero we should first factor to see if we can cancel anything out first:

$\lim\limits_{x\to-\frac{1}{2}^+}\dfrac{(3x-1)(2x+1)}{(2x-3)(2x+1)}$ We can cancel out the 2x + 1.

$\lim\limits_{x\to-\frac{1}{2}^+}\dfrac{3x-1}{2x-3}$ Now we can plug in the ½ without getting a zero in the denominator, so no test value is needed.

$$\lim_{x\to-\frac{1}{2}^+}\frac{3x-1}{2x-3}=\frac{3\left(-\frac{1}{2}\right)-1}{2\left(-\frac{1}{2}\right)-3}=\frac{-\frac{3}{2}-1}{-1-3}=\frac{-\frac{5}{2}}{-4}=\frac{5}{8}$$

Limits at Infinity

The next set of problems deal with limits as x approaches infinity or negative infinity. What does this mean? This means we are finding a horizontal asymptote (if it exists). Most of these type of problems will be given in fraction form. The main technique for these is to divide each term in the numerator and denominator by the highest power of x we see in the <u>DENOMINATOR</u>. We also need to know the following special limit:
$\lim\limits_{x\to\infty}\dfrac{c}{x^n}=0$, where $n>0$. (As the bottom increases, the whole fraction decreases.)

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{3x+2}{x-4}$

To solve this we will divide each term in the numerator and denominator by the highest power of x we see in the <u>DENOMINATOR</u>. In this problem the highest power of x in the denominator will be x.

$$\lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{2}{x}}{\frac{x}{x}-\frac{4}{x}}$$

Now we simplify.

$$\lim_{x\to\infty} \frac{3+\frac{2}{x}}{1-\frac{4}{x}}$$

Now we take the limit of each term separately:

$$\frac{\lim\limits_{x\to\infty} 3+\lim\limits_{x\to\infty}\frac{2}{x}}{\lim\limits_{x\to\infty} 1-\lim\limits_{x\to\infty}\frac{4}{x}} = \frac{3+0}{1-0} = 3$$

So $\lim\limits_{x\to\infty} \dfrac{3x+2}{x-4} = 3$. We can say the horizontal asymptote of $f(x) = \dfrac{3x+2}{x-4}$ is y = 3.

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

The highest power of x in the denominator is x^3, so we divide everything in the top and bottom by x^3.

$$\lim_{x\to\infty} \frac{\frac{3}{x^3}-\frac{2x}{x^3}}{\frac{3x^3}{x^3}-\frac{1}{x^3}}$$

Now simplify.

$$\lim_{x\to\infty} \frac{\frac{3}{x^3}-\frac{2}{x^2}}{3-\frac{1}{x^3}}$$

We will take the limit of each term individually.

$$\frac{\lim\limits_{x\to\infty}\frac{3}{x^3}-\lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty} 3-\lim\limits_{x\to\infty}\frac{1}{x^3}} = \frac{0-0}{3-0} = 0$$

So $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1} = 0$.

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

The highest power of x in the denominator is x , so we divide everything in the top and bottom by x.

$$\lim_{x\to\infty} \frac{\frac{3}{x}-\frac{2x^2}{x}}{\frac{3x}{x}-\frac{1}{x}}$$ Now simplify.

$$\lim_{x\to\infty} \frac{\frac{3}{x}-2x}{3-\frac{1}{x}}$$ We will take the limit of each term individually.

$$\frac{\lim\limits_{x\to\infty}\frac{3}{x}-\lim\limits_{x\to\infty}2x}{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{1}{x}} = \frac{0-\infty}{3-0} = \frac{-\infty}{3} = -\infty$$ Negative infinity divided by any number is still negative infinity.

So $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1} = -\infty$.

EXAMPLE: Find the horizontal asymptote: $y = \dfrac{5x^{\frac{3}{2}}}{4x^{\frac{3}{2}}+1}$.

When you find a horizontal asymptote, it is asking you for what the graph approaches as x gets really big or really small. So in actuality, we are finding the following limit: $\lim\limits_{x\to\infty} \dfrac{5x^{\frac{3}{2}}}{4x^{\frac{3}{2}}+1}$.

The highest power of x in the denominator is $x^{\frac{3}{2}}$, so we divide everything in the top and bottom by $x^{\frac{3}{2}}$.

$$\lim_{x\to\infty} \frac{\frac{5x^{\frac{3}{2}}}{x^{\frac{3}{2}}}}{\frac{4x^{\frac{3}{2}}}{x^{\frac{3}{2}}}+\frac{1}{x^{\frac{3}{2}}}} = \lim_{x\to\infty} \frac{5}{4+\frac{1}{x^{\frac{3}{2}}}}$$ Now we will take the limit of each term individually.

$$\frac{\lim\limits_{x\to\infty}5}{\lim\limits_{x\to\infty}4+\lim\limits_{x\to\infty}\frac{1}{x^{\frac{3}{2}}}} = \frac{5}{4+0} = \frac{5}{4}.$$ So the equation for the horizontal asymptote is: $y = \dfrac{5}{4}$.

EXAMPLE: Find the limit: $\lim\limits_{x\to-\infty} \dfrac{5x^{\frac{3}{2}}}{4\sqrt{x}+1}$

The highest power of x in the denominator is $x^{\frac{1}{2}}$, so we divide everything in the top and bottom by $x^{\frac{1}{2}}$.

$$\lim_{x\to-\infty} \frac{\dfrac{5x^{\frac{3}{2}}}{x^{\frac{1}{2}}}}{\dfrac{4x^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\dfrac{1}{x^{\frac{1}{2}}}}$$ Now simplify

$$\lim_{x\to-\infty} \frac{5x}{4+\dfrac{1}{x^{\frac{1}{2}}}}$$ We will take the limit of each term individually.

$$\frac{\lim\limits_{x\to-\infty} 5x}{\lim\limits_{x\to-\infty} 4+\lim\limits_{x\to-\infty} \dfrac{1}{x^{\frac{1}{2}}}} = \frac{5(-\infty)}{4+0} = \frac{-\infty}{4} = -\infty .$$

So $\lim\limits_{x\to-\infty} \dfrac{5x^{\frac{3}{2}}}{4\sqrt{x}+1} = -\infty$.

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$

For this one, you may think the highest power of x in the denominator is x^2, however we really have $\sqrt{x^2}$ which is really just x. If we start with the identity x = x and we square both sides we will get $x^2 = x^2$. Now take the square root of both sides and we will get $x = \pm\sqrt{x^2}$. So here is the rule. If the limit is going to positive infinity, use the definition $x = \sqrt{x^2}$. If the limit is going to negative infinity, use $x = -\sqrt{x^2}$.

For this example, we will let $x = \sqrt{x^2}$. Since the top is just x we will divide the top by just x. On the bottom we have a square root, so divide this by $\sqrt{x^2}$.

$$\lim_{x\to\infty} \frac{\dfrac{x}{x}}{\dfrac{\sqrt{x^2+1}}{\sqrt{x^2}}}$$ Now simplify.

$$\lim_{x\to\infty} \frac{1}{\sqrt{\dfrac{x^2+1}{x^2}}}$$ We can make this a single square root on the bottom.

$$\lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}$$ We can break up the fraction and then simplify.

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$ Now take the limit of each term individually.

$$\frac{\lim_{x\to\infty} 1}{\sqrt{\lim_{x\to\infty} 1+\lim_{x\to\infty} \frac{1}{x^2}}}=\frac{1}{\sqrt{1+0}}=1$$ So $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}=1$.

EXAMPLE: Find the limit: $\lim_{x\to-\infty} \frac{-3x+1}{\sqrt{x^2+x}}$

Again the highest power in the denominator is x, but this time we will let $x=-\sqrt{x^2}$. We want to divide everything on the top by x and everything on the bottom by $-\sqrt{x^2}$.

$$\lim_{x\to-\infty} \frac{\frac{-3x}{x}+\frac{1}{x}}{\frac{\sqrt{x^2+x}}{-\sqrt{x^2}}}$$ We can simplify this.

$$\lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{-\sqrt{\frac{x^2+x}{x^2}}}$$ We made a single root in the denominator. Now separate the fraction and finish.

$$\lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{-\sqrt{1+\frac{1}{x}}}=\frac{\lim_{x\to-\infty} -3+\lim_{x\to-\infty} \frac{1}{x}}{-\sqrt{\lim_{x\to-\infty} 1+\lim_{x\to-\infty} \frac{1}{x}}}=\frac{-3+0}{-\sqrt{1+0}}=3,$$ so $\lim_{x\to-\infty} \frac{-3x+1}{\sqrt{x^2+x}}=3$

Special Limits with Sine and Cosine

$\lim\limits_{x\to\infty} \sin x = DNE$ $\qquad \lim\limits_{x\to\infty} \cos x = DNE$ $\qquad \lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$ $\qquad \lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{x - \cos x}{x}$

$\lim\limits_{x\to\infty} \dfrac{x}{x} - \dfrac{\cos x}{x}$ First we will separate the fraction

$\lim\limits_{x\to\infty} 1 - \dfrac{\cos x}{x}$ After simplifying, take the limit of each term separately.

$\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \dfrac{\cos x}{x}$ Calculate the limit

$1 - 0 = 1$ So $\lim\limits_{x\to\infty} \dfrac{x - \cos x}{x} = 1$

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \cos\left(\dfrac{1}{x}\right)$

We know that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, so we will have $\lim\limits_{x\to\infty} \cos\left(\dfrac{1}{x}\right) = \cos(0) = 1$.

EXAMPLE: Find the limit: $\lim\limits_{x\to\infty} \dfrac{4}{3x - \sin x}$

First divide the top and bottom by x since that is the highest power in the denominator.

$\lim\limits_{x\to\infty} \dfrac{\frac{4}{x}}{\frac{3x}{x} - \frac{\sin x}{x}}$ Now simplify and take the limit of each term individually.

$$\lim_{x\to\infty} \frac{\frac{4}{x}}{3 - \frac{\sin x}{x}} = \frac{\lim\limits_{x\to\infty} \frac{4}{x}}{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \frac{\sin x}{x}} = \frac{0}{3 - 0} = 0$$