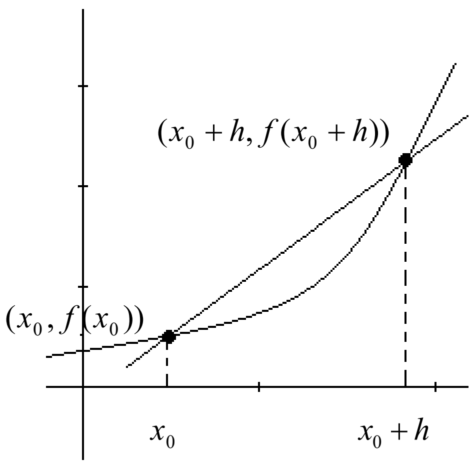
3.1 Defining the Derivative

You will see that a derivative is just a fancy word for a slope. We can easily find the slope at a certain point if we have a straight line, but what about curved lines? To find this we will need to review the difference quotient, which is something you hopefully saw in precalculus. We will be estimating the slope of a tangent line to a curve f(x) at point $x = x_0$ by choosing two points that are close together. The first point will be at $x = x_0$. The second point will be very close to $x = x_0$. We will go a little bit past x_0 and we will call this amount Δx. In the picture below we will write the coordinates of each point using these variables:



If we want to find the slope of the line through the two points, we will need to use the slope formula, which is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Using our notation we get: $m = \dfrac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$. This simplifies to: $m = \dfrac{f(x_0 + h) - f(x_0)}{h}$. This is the difference quotient. Now we want to find the slope right at point x_0, so in order to this we will make h so small that both points are on top of each other at x_0. So we want h to go to zero. Sounds like a limit to me! This is what we are missing. So now we have our definition.

Definition for the slope of a tangent line at $x = x_0$: $\quad f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

EXAMPLE: Find the equation of the tangent line at the point (2, 1) on the curve $f(x) = 5 - x^2$.

We will use the definition above. The x_0 is always the x coordinate of the point they give you. So $x_0 = 2$.

$\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ Now that we put in a 2 for h we need to find each piece of the formula. Let's first start with $f(2)$. Put a 2 in for x in the $f(x)$ equation. You will get $f(2) = 5 - 2^2$, so $f(2) = 1$. Now let's find $f(2+h)$. To do this replace the x in the $f(x)$ equation with $2 + h$. You will get the following:

$f(2+h) = 5 - (2+h)^2$ Now distribute and multiply

$f(2+h) = 5 - (2+h)(2+h)$

$f(2+h) = 5 - (h^2 + 4h + 4)$

$f(2+h) = 5 - h^2 - 4h - 4$

$f(2+h) = 1 - h^2 - 4h$

We have found both pieces we need for the formula. We have $f(2+h) = 1 - h^2 - 4h$ and $f(2) = 1$. replace the $f(2+h)$ with $1 - h^2 - 4h$ and we will replace the $f(2)$ with a 1. Then you will have:

$\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} \dfrac{(1 - h^2 - 4h) - 1}{h}$. Now we need to simplify this and factor the result.

$\lim\limits_{h\to 0}\dfrac{-h^2-4h}{h}=\lim\limits_{h\to 0}\dfrac{h(-h-4)}{h}=\lim\limits_{h\to 0}(-h-4)$. Now put in a 0 for h and you will get -4.

So this means that the slope of the tangent line at x = 2 is -4. We can also say that $f'(2)=-4$. Now we need to find the equation of the tangent line. We will start with the equation $y = mx + b$. We will know that $m=f'(2)=-4$, $x = 2$, and $y = 1$. Plug these into the formula: 1 = -4(2) + b. Now solve for b. You will get b = 9. So our equation is $y = -4x + 9$.

EXAMPLE: Find the equation of the tangent line at the point (-1, -9) on the curve $g(t)=t^3-8$.

So this time $x_0=-1$. If we put that into the derivative formula and change the f(x) to g(t) we will get:

$g'(t)=\lim\limits_{h\to 0}\dfrac{g(-1+h)-g(-1)}{h}$ Now we will find $g(-1+h)$ and $g(-1)$. We will get:

$g(-1)=(-1)^3-8=-1-8=-9$ and we also have:
$g(-1+h)=(h-1)^3-8=\left(h^3-3h^2+3h-1\right)-8=h^3-3h^2+3h-9$. We will put these into the derivative formula:

$\lim\limits_{h\to 0}\dfrac{h^3-3h^2+3h-9-(-9)}{h}=\lim\limits_{h\to 0}\dfrac{h^3-3h^2+3h}{h}=\lim\limits_{h\to 0}\dfrac{h(h^2-3h+3)}{h}=\lim\limits_{h\to 0}h^2-3h+3=0-0+3=3$. So the slope of tangent line to this curve at this point is 3. We can say that $g'(-1)=3$. Now we need to find the equation of the tangent line. We will start with the equation y = mx + b. We will know that $m=g'(-1)=3$, x = -1, and y = -9. Plug these into the formula: -9 = 3(-1) + b. Now solve for b. You will get b = -6. So our equation is $y = 3x - 6$.

EXAMPLE: Find the equation of the tangent line at the point (2, 2) on the curve $f(x)=\dfrac{8}{x^2}$.

So this time $x_0=2$. If we put that into the derivative formula we will get:

$f'(2)=\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$ Now we will find $f(2+h)$ and $f(2)$. We will get:

$f(2)=\dfrac{8}{2^2}=2$ and we also have:
$f(2+h)=\dfrac{8}{(2+h)^2}$. We will put these into the derivative formula:

$$\lim_{h\to 0}\frac{\frac{8}{(2+h)^2}-\frac{2}{1}}{h}=\lim_{h\to 0}\frac{\frac{8-2(2+h)^2}{(2+h)^2}}{h}=\lim_{h\to 0}\frac{8-2(4+4h+h^2)}{h(2+h)^2}=\lim_{h\to 0}\frac{8-8-8h-2h^2}{h(2+h)^2}=\lim_{h\to 0}\frac{h(-8-2h)}{h(2+h)^2}.$$ So

$\lim_{h\to 0}\frac{(-8-2h)}{(2+h)^2}=\frac{-8-0}{(2+0)^2}=-2$. So the slope of tangent line to this curve at this point is 2. We can say that $f'(2)=-2$. Now we need to find the equation of the tangent line. We will start with the equation $y = mx + b$. We will know that $m=f'(2)=-2$, $x = 2$, and $y = 2$. Plug these into the formula: $2 = -2(2) + b$. Now solve for b. You will get $b = 6$. So our equation is $y = -2x + 6$.

EXAMPLE: Find the equation of the tangent line at the point (8, 3) on the curve $f(x)=\sqrt{x+1}$.

So this time $x_0=8$. If we put that into the derivative formula we will get:

$f'(8)=\lim_{h\to 0}\frac{f(8+h)-f(8)}{h}$ Now we will find $f(8+h)$ and $f(8)$. We will get:

$f(8)=\sqrt{8+1}=\sqrt{9}=3$ and we also have:
$f(8+h)=\sqrt{8+h+1}=\sqrt{9+h}$. We will put these into the derivative formula:

$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$. Now in order to solve this one we need to multiply top and bottom by the conjugate.

$$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim_{h\to 0}\frac{9+h+3\sqrt{9+h}-3\sqrt{9+h}-9}{h\sqrt{9+h}+3}=\lim_{h\to 0}\frac{h}{h\sqrt{9+h}+3}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}.$$ Now put in a zero for h to get: $\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{\sqrt{9+0}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$.

So the slope of tangent line to this curve at this point is 1/6. We can say that $f'(8)=\frac{1}{6}$. Now we need to find the equation of the tangent line. We will start with the equation y = mx + b. We now know that: $m=f'(8)=\frac{1}{6}$, $x = 8$, and $y = 3$. Plug these into the formula: $3=\frac{1}{6}(8)+b$. So this simplifies to $3=\frac{4}{3}+b$. Now add the fraction to both sides: $3-\frac{4}{3}=b$. We will get $b=\frac{5}{3}$. So our equation is: $y=\frac{1}{6}x+\frac{5}{3}$.

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EXAMPLE: Find the value of $\left.\frac{dy}{dx}\right|_{x=3}$ if $f(x)=x^2-3x$. Then use it to find the equation of the tangent line at this value of x.

The notation $\left.\frac{dy}{dx}\right|_{x=3}$ means $f'(3)$. These are two different ways that we can express the derivative at a certain value of x.

We will now proceed as we did with the previous problems.

$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$ Now we will find $f(3+h)$ and $f(3)$. We will get:

$f(3)=3^2-3(3)=0$ and we also have:
$f(3+h)=(3+h)^2-3(3+h)$
$f(3+h)=(3+h)(3+h)-3(3+h)$
$f(3+h)=9+6h+h^2-9-3h$
$f(3+h)=h^2+3h$

Now let's put this into the derivative formula.

$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$f'(3)=\lim_{h\to 0}\frac{h^2+3h-0}{h}$ Now simplify.

$f'(3)=\lim_{h\to 0}\frac{h^2+3h}{h}$ Now factor out an h.

$f'(3)=\lim_{h\to 0}\frac{h(h+3)}{h}$ Cancel out the h terms from the top and bottom.

$f'(3)=\lim_{h\to 0}h+3$ Finally put in a 0 for h.
$f'(3)=0+3$. So we have $f'(3)=3$.

To find our equation we need to find our point to put into $y = mx + b$. We will use the original equation for this one: $f(3)=3^2-3(3)=0$. So our point is (3, 0). Now we plug our values into $y = mx + b$: $0 = 3(3) + b$
Solving will give us $b = -9$. So our equation is $y = 3x - 9$.

EXAMPLE: At what point(s) does the graph of $f(x)=3x^2-6x+1$ have a horizontal tangent?

In order to do this, we first need to find the derivative using the formula $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. However in this case we are not given x_0, so instead we will use *x*: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

We will do this the same way as above. First we will find $f(x+h)$.

$f(x+h)=3(x+h)^2-6(x+h)+1$
$f(x+h)=3(x+h)(x+h)-6x-6h+1$
$f(x+h)=3(x^2+2xh+h^2)-6x-6h+1$
$f(x+h)=3x^2+6xh+3h^2-6x-6h+1$

Now that we have simplified this as much as possible, we will put it into the derivative formula.

$\lim_{h\to 0}\frac{3x^2+6xh+3h^2-6x-6h+1-(3x^2-6x+1)}{h}$ Now we will distribute the minus into $f(x)$.

$\lim_{h\to 0}\frac{3x^2+6xh+3h^2-6x-6h+1-3x^2+6x-1}{h}$ Now cancel and simplify.

$\lim_{h\to 0}\frac{6xh+3h^2-6h}{h}$ Now we can factor out an h from the top.

$\lim_{h\to 0}\frac{h(6x+3h-6)}{h}$ We cancel the h from top and bottom.

$\lim_{h\to 0}6x+3h-6$ Now plug in 0 for h. You will get $6x-6$.

So $6x-6$ is our derivative. This is an expression where we can find the slope of $f(x)$ at any value of x. If you have a horizontal tangent, then this means that the slope is zero. We will set the derivative equal to zero. So $6x-6=0$. You will get $x=1$. The question asked us for which point is there a horizontal tangent, so we need the y value: $f(1)=3(1)^2-6(1)+1=-2$. So our point is (1, -2). What does this point represent? This is actually the vertex of the parabola. In Precalculus you probably learned the vertex formula. Now you have seen how to find the vertex using Calculus.