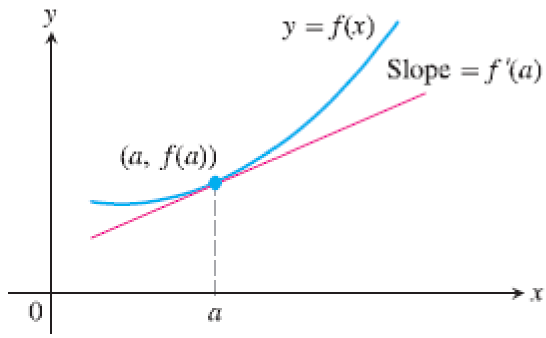
3.11 Linear Approximations and Differentials

Sometimes we can approximate more complicated functions with simpler ones. These would give us enough accuracy for certain situations and are easier to work with. The approximating functions in this section are called **linearizations**, and they are based on tangent lines. It is possible to have approximating functions that are polynomials, but we will not discuss this here.

Given the following figure, we want to come up with and equation for our tangent line. As we zoom in, the closer the tangent line will lay on top of the original curve allowing us to find the exact slope at the value of $x = a$. In order to find the equation for the tangent line we will start with the point-slope formula: $y - y_1 = m(x - x_1)$. Here the m (slope) is $f'(a)$. Also $x_1 = a$ and $y_1 = f(a)$. We will substitute both of these into the point-slope formula to get:



$y - f(a) = f'(a)(x - a)$ We will now solve for y.
$y = f(a) + f'(a)(x - a)$ The y is expressed as $L(x)$.
$L(x) = f(a) + f'(a)(x - a)$ This is the linearization of f at a.

If f is differentiable at $x = a$, then the approximating function is $L(x) = f(a) + f'(a)(x - a)$ is the linearization of f at a. The approximation $f(x) \approx L(x)$ of f by L is the standard linearization of f at a. The point $x = a$ is the center of the linearization.

EXAMPLE: Find the linearization $L(x)$ of $f(x) = \sqrt{x^2 + 16}$ at $x = -3$.

So we will be using $L(x) = f(a) + f'(a)(x - a)$. So we need to find the individual components of this formula and substitute it all in. Let's first find $f(a)$: $f(3) = \sqrt{(-3)^2 + 16} = \sqrt{9 + 16} = 5$. Then we need to find $f'(a)$. In order to do this we will first we will find the derivative: $f'(x) = \frac{1}{2}\left(x^2 + 16\right)^{-\frac{1}{2}}(2x)$. This can be simplified to: $f'(x) = \frac{x}{\sqrt{x^2 + 16}}$. Now we will find $f'(a)$: $f'(-3) = \frac{-3}{\sqrt{(-3)^2 + 16}} = \frac{-3}{5}$. Now we will substitute everything into the linearization formula: $L(x) = f(a) + f'(a)(x - a)$:

$L(x) = 5 + \frac{-3}{5}(x - (-3))$

$L(x) = 5 - \frac{3}{5}(x + 3)$

$L(x) = 5 - \frac{3}{5}x - \frac{9}{5}$. Simplifying gives us $L(x) = -\frac{3}{5}x + \frac{16}{5}$ which is our final answer.

EXAMPLE: Find the linearization $L(x)$ of $f(x)=\sin x$ at $x=\frac{\pi}{6}$.

So we will be using $L(x)=f(a)+f'(a)(x-a)$. So we need to find the individual components of this formula and substitute it all in. Let's first find $f(a)$: $f\left(\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$. Then we need to find $f'(a)$. In order to do this we will first we will find the derivative: $f'(x)=\cos x$. Now we will find $f'(a)$: $f'\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$.
Now we will substitute everything into the linearization formula: $L(x)=f(a)+f'(a)(x-a)$:

$$L(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)$$
$$L(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}x-\frac{\pi\sqrt{3}}{12}$$
$$L(x)=\frac{\sqrt{3}}{2}x+\frac{1}{2}-\frac{\pi\sqrt{3}}{12}$$
$$L(x)=\frac{\sqrt{3}}{2}x+\frac{6-\pi\sqrt{3}}{12}$$

EXAMPLE: Given $x_0=-0.9$, find a linearization that will replace $f(x)=3x^2+2x-4$ by centering your linearization not on x_0, but at a nearby integer $x=a$. This will make the derivative and function easy to evaluate.

We want to find an integer that is closest to x_0, so in this case we will let $a=-1$. We will now use $L(x)=f(a)+f'(a)(x-a)$. So we need to find the individual components of this formula and substitute it all in. Let's first find $f(a)$: $f(-1)=3(-1)^2+2(-1)-5=-3$. Then we need to find $f'(a)$. In order to do this we will first we will find the derivative: $f'(x)=6x+2$. Now we will find $f'(a)$: $f'(-1)=6(-1)+2=-4$. Now we will substitute everything into the linearization formula: $L(x)=f(a)+f'(a)(x-a)$:

$$L(x)=-3+-4(x-(-1))$$
$$L(x)=-3-4(x+1)$$
$$L(x)=-3-4x-4$$
$$L(x)=-4x-7$$

EXAMPLE: Given $x_0=\frac{11}{5}$, find a linearization that will replace $f(x)=\frac{2x}{x-1}$ by centering your linearization not on x_0, but at a nearby integer $x=a$. This will make the derivative and function easy to evaluate.

We want to find an integer that is closest to x_0, so in this case we will let $a=2$. We will now use $L(x)=f(a)+f'(a)(x-a)$. So we need to find the individual components of this formula and substitute it all in. Let's first find $f(a)$: $f(2)=\frac{2(2)}{2-1}=4$. Then we need to find $f'(a)$. In order to do this we will first we will find the derivative.   This will involve the quotient rule: $f'(x)=\frac{(x-1)(2)-2x(1)}{(x-1)^2}=\frac{2x-2-2x}{(x-1)^2}=\frac{-2}{(x-1)^2}$.

Now we will find $f'(a)$: $f'(2)=\frac{-2}{(2-1)^2}=\frac{-2}{1^2}=-2$. Now we will substitute everything into the linearization formula $L(x)=f(a)+f'(a)(x-a)$:

$L(x)=4+-2(x-2))$
$L(x)=4-2x+4$
$L(x)=-2x+8$

EXAMPLE: Given $x_0=\frac{\pi}{12}$, find a linearization that will replace $f(x)=\tan^{-1}x$ by centering your linearization not on x_0, but at a nearby integer $x=a$. This will make the derivative and function easy to evaluate.

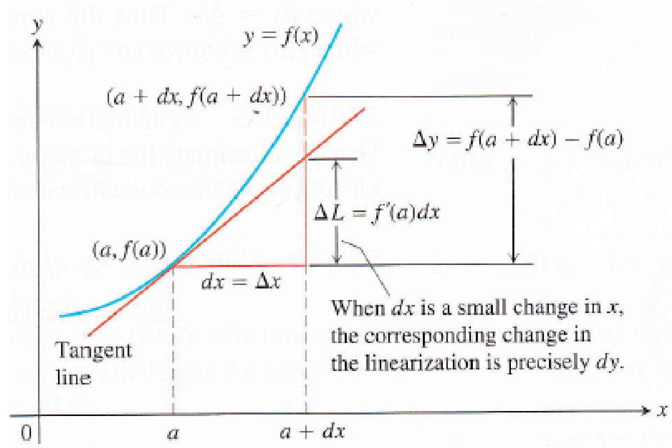
We want to find an integer that is closest to x_0, so in this case we will let $a=0$. We will now use $L(x)=f(a)+f'(a)(x-a)$. So we need to find the individual components of this formula and substitute it all in. Let's first find $f(a)$: $f(0)=\tan^{-1}0=0$. Then we need to find $f'(a)$. In order to do this we will first we will find the derivative: $f'(x)=\frac{1}{1+x^2}$. Now we will find $f'(a)$: $f'(0)=\frac{1}{1+0^2}=1$. Now we will substitute everything into the linearization formula: $L(x)=f(a)+f'(a)(x-a)$:

$L(x)=0+1(x-0)$
$L(x)=x$ This is our final answer.

Differentials

Back in the section regarding implicit differentiation, we were solving for $\frac{dy}{dx}$. We treated this as one variable. However in this section we will consider this made up of two different variables, *dy* and *dx*. Let $y=f(x)$ be a differentiable function. The **differential *dx*** is an independent variable. The **differential *dy*** is $dy=f'(x)dx$. But what does this mean in terms of something physical?   The below diagram illustrates this geometrically.

Let $x=a$ and set $dx=\Delta x$. The corresponding change in $y=f(x)$ is:
$\Delta y=f(a+dx)-f(a)$. The corresponding change in the tangent line L is:
$\Delta L=L(a+dx)-L(a)=f(a)+f'(a)[(a+dx)-a]-f(a)$
$=f'(a)dx$
Therefore the change in the linearization of *f* is precisely the value of the differential *dy* when $x=a$ and $dx=\Delta x$. So the *dy* represents the amount the tangent line rises or falls when *x* changes by an amount $dx=\Delta x$.



EXAMPLE: Find *dy* given $y = x^4 - 6\sqrt{x}$.

First we will find the derivative $\frac{dy}{dx}$. We are using this notation since we are working with differentials. So, $\frac{dy}{dx} = 4x^3 - 6 \cdot \frac{1}{2} x^{-\frac{1}{2}}$. This simplifies to $\frac{dy}{dx} = 4x^3 - \frac{3}{\sqrt{x}}$. The *dy* and *dx* can be thought of as two separate variables, so multiply both sides by dx in order to isolate the *dy*.   You will get the final answer, which is $dy = \left(4x^3 - \frac{3}{\sqrt{x}}\right)dx$.

EXAMPLE: Find *dy* given $xy^3 - 8x^{\frac{3}{2}} - y = 0$.

This one involves implicit differentiation. The first term xy^3 involves the product rule:

$x \cdot 3y^2 \frac{dy}{dx} + y^3(1) - 8 \cdot \frac{3}{2} x^{\frac{1}{2}} - \frac{dy}{dx} = 0$.   Now simplify

$3xy^2 \frac{dy}{dx} + y^3 - 12\sqrt{x} - \frac{dy}{dx} = 0$ Get all terms with $\frac{dy}{dx}$ on one side of the equation

$3xy^2 \frac{dy}{dx} - \frac{dy}{dx} = 12\sqrt{x} - y^3$ Factor out the $\frac{dy}{dx}$.

$\frac{dy}{dx}\left(3xy^2 - 1\right) = 12\sqrt{x} - y^3$ Now solve for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{12\sqrt{x} - y^3}{3xy^2 - 1}$ Now multiply both sides by *dx*.

$dy = \left(\frac{12\sqrt{x} - y^3}{3xy^2 - 1}\right)dx$ It is now solved for *dy*, so this is our answer.

EXAMPLE: Estimate the volume of material in a cone-shaped shell with a height of 20 in, radius 9 in, and shell thickness 0.5 in.

First we start with the formula for a cone: $V = \frac{1}{3}\pi r^2 h$. Then we want to take the derivative with respect to r: $\frac{dV}{dr} = \frac{1}{3} \cdot 2\pi rh$. Notice that since we are taking the derivative with respect to *r*, the *h* is treated like a constant. Now we will multiply both sides by dr: $dV = \left(\frac{2}{3}\pi rh\right)dr$. Now we plug in our given information. We are given that *h* = 20 in, *r* = 9 in, and *dr* = 0.5 in:

$$dV = \left(\frac{2}{3}\pi(9)(20)\right)(0.5)$$

$$dV = 60\pi \approx 188.4 \ \ in^3$$

EXAMPLE: The radius of a sphere is measured as 6 cm with an error of 3%. The volume of the sphere is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

To estimate the percentage error in the volume calculation, we will divide the estimated change in volume due to error by the actual volume of the sphere with the given radius.

First let's find the change in volume due to error. This is the same as dV. First we start with the formula for a sphere: $V = \frac{4}{3}\pi r^3$. Next we will take the derivative of both sides of our volume formula with respect to r: $\frac{dV}{dr} = \frac{4}{3} \cdot 3\pi r^2$. This simplifies to: $\frac{dV}{dr} = 4\pi r^2$. If we multiply both sides by dr we get: $dV = 4\pi r^2 dr$. We need to know which information to plug into this. To find dr, we will multiply the measurement of the radius by the error, which in this case is 3%. So $dr = 6(0.03) = 0.18$. Now we can plug in the values to find dV: $dV = 4\pi(6)^2(0.18) = 25.92\pi\ cm^3$.

Next, let's find the actual volume of the sphere with the given radius: $V = \frac{4}{3}\pi(6)^3 = 288\pi\ cm^3$.

So to find the percentage error in the volume calculation, divide: $\frac{25.92\pi}{288\pi} = 0.09 = 9\%$.