# 3.2 The Derivative as a Function

**Derivative of a function for any value x**

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The above equation is the general one to find an expression to get the slope at any x value. Notice the notation that is being used to indicate a derivative: $f'(x)$. This process is called finding the derivative by the limit process. We will look at other ways to find derivatives in later sections.

EXAMPLE: Find the derivative of $f(x) = 1 - x^2$ by the limit process. Then use your answer to find: $f'(-3)$, $f'(0)$, and $f'\left(\frac{1}{2}\right)$.

This is similar to the previous examples except this time we don't have a $x_0$. We still want to find each part of the formula above. We already have $f(x)$. Now we need to find $f(x+h)$.

$f(x+h) = 1-(x+h)^2$
$f(x+h) = 1-(x+h)(x+h)$
$f(x+h) = 1-(x^2+2xh+h^2)$
$f(x+h) = 1-x^2-2xh-h^2$

Now that we have $f(x+h)$ and $f(x)$ we can put them into the derivative formula.

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0}\frac{(1-x^2-2xh-h^2)-(1-x^2)}{h}$ Here, $f(x+h) = 1-x^2-2xh-h^2$ and $f(x) = 1-x^2$.

$f'(x) = \lim_{h\to 0}\frac{1-x^2-2xh-h^2-1+x^2}{h}$ We will distribute the minus sign and then simplify the numerator.

$f'(x) = \lim_{h\to 0}\frac{-2xh-h^2}{h}$ In order to do the limit we must factor out the h and cancel it out.

$f'(x) = \lim_{h\to 0}\frac{h(-2x-h)}{h} = \lim_{h\to 0} -2-h$. Now that we canceled, we can replace the h with a 0.

$f'(x) = \lim_{h\to 0} -2x-h = -2-0 = -2x$ So we write $f'(x) = -2x$. This is an expression that will allow us to find the slope of f at any value of x.

Now let's use our answer to find $f'(-3)$. We will put in a -3 into the equation -2$x$ to get: $f'(-3) = -2(-3) = 6$.

We do the same for the others: $f'(0) = -2(0) = 0$. $f'\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right) = -1$

EXAMPLE: Find the derivative of $f(x) = -5$ by the limit process. Then use your answer to find: $f'(-100)$

This one is a lot easier because there are no variables. What is $f(x+h)$? Since there are no variables to plug $x+h$ into then $f(x+h) = -5$. So in the derivative formula we'll put in a -5 for both $f(x+h)$ and $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = \lim_{h \to 0} \frac{-5-(-5)}{h} = \lim_{h \to 0} \frac{0}{h}$ Now we will get a zero on top causing the whole thing to be zero.

$f'(x) = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$ So $f'(x) = 0$. This will happen whenever we are taking the slope of a single number with no variables. Let's think about why this happens. $f(x) = -5$ is a horizontal line crossing the y-axis at -5. Horizontal lines are not increasing or decreasing, so the slope should be zero.

When we find $f'(-100)$, the answer would be zero since at any x value the slope is always zero.

EXAMPLE: Find the derivative of $f(x) = \sqrt{x} + 6$ by the limit process.

This one is going to be done a little different. First we need to find $f(x+h)$.

$f(x+h) = \sqrt{x+h} + 6$ We just put in a $x+h$ for x in our original equation for f. We are ready to substitute this information into our derivative formula.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} + 6 - (\sqrt{x} + 6)}{h}$ Here, $f(x+h) = \sqrt{x+h} + 6$ and $f(x) = \sqrt{x} + 6$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} + 6 - \sqrt{x} - 6}{h}$ Make sure you remember to distribute the negative and cancel the 6 & -6.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ Okay so now we need to multiply top and bottom by the conjugate of the top.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$ This was the same technique used in the last chapter.

$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h(\sqrt{x+h} + \sqrt{x})}$ We will apply the difference of squares when multiplying on top.

$f'(x) = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$ So not the square roots cancelled, so we now cancel the $x$ and $-x$.

$f'(x) = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$ Now we can cancel the $h$ from the top and bottom.

$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$ Now we can put in a 0 for $h$, simplify, and we are done.

$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. So our final answer is $f'(x) = \frac{1}{2\sqrt{x}}$.

EXAMPLE: Find the derivative of $f(x)=\dfrac{4}{\sqrt{x}}$ by the limit process.

First we will find $f(x+h)$. This will be $f(x+h)=\dfrac{4}{\sqrt{x+h}}$. I will leave it in this form. Now we will put in a $\dfrac{4}{\sqrt{x+h}}$ for $f(x+h)$ and a $\dfrac{4}{\sqrt{x}}$ for $f(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4}{\sqrt{x+h}}-\dfrac{4}{\sqrt{x}}}{h}$$ So now we need common denominators in the numerator.

$$f'(x)=\lim_{h\to 0}\frac{\left(\dfrac{\sqrt{x}}{\sqrt{x}}\right)\dfrac{4}{\sqrt{x+h}}-\dfrac{4}{\sqrt{x}}\left(\dfrac{\sqrt{x+h}}{\sqrt{x+h}}\right)}{h}$$ Multiply each by a power of one.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4\sqrt{x}}{\sqrt{x}\left(\sqrt{x+h}\right)}-\dfrac{4\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}}{h}$$ Now combine to get a single fraction on top.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4\sqrt{x}-4\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}}{\dfrac{h}{1}}$$ Now let's clear the double fractions

$$f'(x)=\lim_{h\to 0}\frac{4\sqrt{x}-4\sqrt{x+h}}{h\sqrt{x}\left(\sqrt{x+h}\right)}$$ Now we need to multiply by the conjugate again.

$$f'(x)=\lim_{h\to 0}\frac{4\sqrt{x}-4\sqrt{x+h}}{h\sqrt{x}\left(\sqrt{x+h}\right)}\cdot\frac{4\sqrt{x}+4\sqrt{x+h}}{4\sqrt{x}+4\sqrt{x+h}}$$ We will use the difference of squares again.

$$f'(x)=\lim_{h\to 0}\frac{\left(4\sqrt{x}\right)^2-\left(4\sqrt{x+h}\right)^2}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now simplify the top.

$$f'(x)=\lim_{h\to 0}\frac{16x-16(x+h)}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now distribute the negative and cancel the 16x terms.

$$f'(x)=\lim_{h\to 0}\frac{-16h}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now cancel the *h* from the top and bottom.

$$f'(x)=\lim_{h\to 0}\frac{-16}{\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ So now we put in a 0 for *h*.

$$f'(x)=\frac{-16}{\sqrt{x}\left(\sqrt{x+0}\right)\left(4\sqrt{x}+4\sqrt{x+0}\right)}=\frac{-16}{\sqrt{x}\sqrt{x}(4\sqrt{x}+4\sqrt{x})}=\frac{-16}{x\left(8\sqrt{x}\right)}=\frac{-16}{8x\sqrt{x}}=\frac{-2}{x\sqrt{x}}$$ So $f'(x)=\dfrac{-2}{x\sqrt{x}}$.

EXAMPLE: Find the derivative of $f(x)=\dfrac{1}{x^2}$ by the limit process.

First we will find $f(x+h)$. This will be $f(x+h)=\dfrac{1}{(x+h)^2}$. I will leave it in this form. Now we will put in a $\dfrac{1}{(x+h)^2}$ for $f(x+\Delta x)$ and a $\dfrac{1}{x^2}$ for $f(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$ So now we need common denominators in the numerator.

$$f'(x)=\lim_{h\to 0}\frac{\left(\frac{x^2}{x^2}\right)\frac{1}{(x+h)^2}-\frac{1}{x^2}\left(\frac{(x+h)^2}{(x+h)^2}\right)}{h}$$ We multiply each fraction by a power of one.

$$f'(x)=\lim_{h\to 0}\frac{\frac{x^2}{x^2(x+h)^2}-\frac{(x+h)^2}{x^2(x+h)^2}}{h}$$ Now we can combine the two top fractions into one.

$$f'(x)=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{\frac{h}{1}}$$ We need to get rid of these double fractions. Multiply by the reciprocal.

$$f'(x)=\lim_{h\to 0}\frac{x^2-(x+h)^2}{(h)x^2(x+h)^2}$$ We need to simplify the numerator by multiplying.

$$f'(x)=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$$ Now distribute the negative and cancel the $x^2$.

$$f'(x)=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}$$ Now we factor out an *h* from the numerator.

$$f'(x)=\lim_{h\to 0}\frac{h(-2x-h)}{hx^2(x+h)^2}$$ We can cancel out the *h* from the top and bottom.

$$f'(x)=\lim_{h\to 0}\frac{(-2x-h)}{x^2(x+h)^2}$$ So now we are finally about to put in a 0 for $h$.

$$f'(x)=\lim_{h\to 0}\frac{(-2x-h)}{x^2(x+h)^2}=\frac{(-2x-0)}{x^2(x+0)^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}.$$ So we finally have our answer: $f'(x)=\dfrac{-2}{x^3}$.

EXAMPLE: Find the value of $\left.\frac{dy}{dx}\right|_{x=3}$ if $f(x)=x^3+2$. Then use it to find the equation of the tangent line at this value of x.

The notation $\left.\frac{dy}{dx}\right|_{x=3}$ is basically saying to find $f'(3)$. So we will proceed to find the derivative by the limit process as before.

First we will find $f(x+h)$. This will be $f(x+h)=(x+h)^3+2$. You will need to multiply this out. This is: $(x+h)(x+h)(x+h)+2$. Multiply two of them and then multiply the result by the last one. You will get: $(x+h)(x^2+2xh+h^2)+2=x^3+3x^2h+3xh^2+h^3+2$. So $f(x+h)=x^3+3x^2h+3xh^2+h^3+2$.
Let's put this into the derivative formula.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+2+h^3-(x^3+2)}{h}$$ Now simplify.

$$f'(x)=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$ Now factor out a $\Delta x$.

$$f'(x)=\lim_{h\to 0}\frac{h\left(3x^2+3xh+h^2\right)}{h}$$ Cancel out the $\Delta x$ terms from the top and bottom.

$f'(x)=\lim\limits_{h\to 0}3x^2+3xh+h^2$ Finally put in a 0 for h.

$f'(x)=3x^2+3x(0)+(0)^2=3x^2$. So we have $f'(x)=3x^2$. We need to put in a 3 from above.

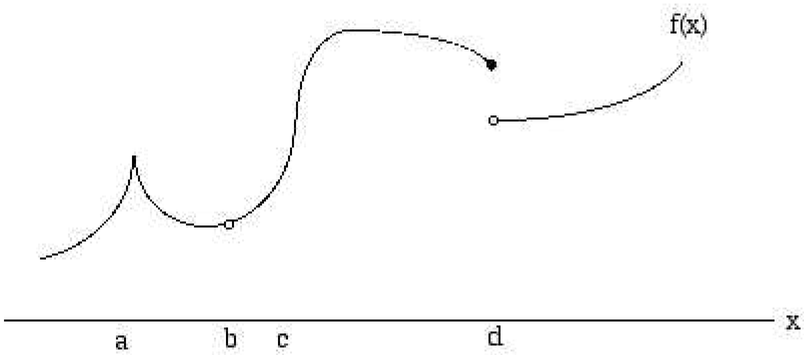
$f'(3)=3(3)^2=27$.

To find our equation we need to find our point to put into $y = mx + b$. We will use the original equation for this one: $f(3)=3^3+2=29$. So our point is (3, 29). Now we plug our values into $y = mx + b$: $29 = 27(3) + b$
Solving will give us $b = 110$. So our equation is $y = 27x + 110$.

More on next page…

**When does a derivative *not* have a derivative at a point?**

Because we find derivatives using limits, what if the limit at a certain point does not exist? This means there is no derivative at that point. In the graph below, there are four points where the derivative does not exist.




**At point a**: There is a cusp (or corner) so the derivative does not exist at a.

**At point b**: There is a hole in the graph, so the derivative does not exist at b.

**At point c**: There is a vertical tangent here, so the derivative does not exist at c.

**At point d**: There is a discontinuity at d (gap in graph), so the derivative does not exist at d.

EXAMPLE: Find $f'(1)$ if possible given $f(x)=\begin{cases} 2 & if \;\; x\le 1 \\ 2x & if \;\; x>1 \end{cases}$.

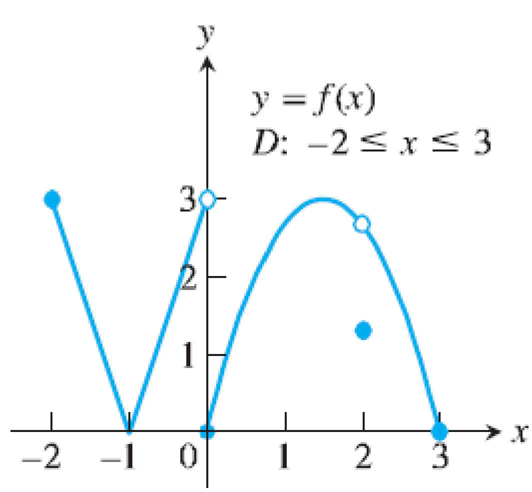
Notice that this would be continuous at $x = 1$ since the $y$-values of each function match at $x = 1$. I will take the derivative of each piece separately. NOTE: When $h < 0$, $1 + h < 1$ so we use the top function: $f(1 + h) = 2$. When $h > 0$, $1 + h > 1$, so we use bottom function: $f(1 + h) = 2(1 + h)$.

First we find the derivative of $f(x)=2$: $\lim\limits_{h\to 0^-}\dfrac{f(1+h)-f(1)}{h}=\lim\limits_{h\to 0^-}\dfrac{2-2}{h}=\lim\limits_{h\to 0^-}0=0$

Now the derivative of $f(x)=2x$: $\lim\limits_{h\to 0^+}\dfrac{f(1+h)-f(1)}{h}=\lim\limits_{h\to 0^+}\dfrac{2(1+h)-2}{h}=\lim\limits_{h\to 0^+}\dfrac{2+2h-2}{h}=\lim\limits_{h\to 0^+}\dfrac{2h}{h}=2$

As you can see here the left and right side limits do match, so therefore the limit does not exist. This means the derivative at the $x$ value of 1 also does not exist.

EXAMPLE: The figure below shows a graph of a function over the closed interval [-2, 3]. At what domain points does the graph appear to be: a.) differentiable b.) continuous but not differentiable? c.) neither continuous or differentiable?




a.) First let's see where the graph is not differentiable based on what was mentioned above. The derivative does not exist at -1 (corner), 0 (not continuous), and at 2 (hole). Therefore we want to indicate all x values besides these. The answer is: $[-2,\ -1)\cup(-1,\ 0)\cup(0,\ 2)\cup(2,\ 3]$.

b.) At x = -1 the graph would be continuous (no break) however the derivative would not exist here because there is a corner.

c.) At $x = 0$ and $x = 2$ the graph is not continuous, so because of this derivative does not exist here either.