3.2 The Derivative as a Function

Derivative of a function for any value x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The above equation is the general one to find an expression to get the slope at any x value. Notice the notation that is being used to indicate a derivative: $f'(x)$. This process is called finding the derivative by the limit process. We will look at other ways to find derivatives in later sections.

EXAMPLE: Find the derivative of $f(x) = 1 - x^2$ by the limit process. Then use your answer to find: $f'(-3)$, $f'(0)$, and $f'\left(\frac{1}{2}\right)$.

This is similar to the previous examples except this time we don't have a x_0. We still want to find each part of the formula above. We already have $f(x)$. Now we need to find $f(x+h)$.

$f(x+h) = 1 - (x+h)^2$
$f(x+h) = 1 - (x+h)(x+h)$
$f(x+h) = 1 - (x^2 + 2xh + h^2)$
$f(x+h) = 1 - x^2 - 2xh - h^2$

Now that we have $f(x+h)$ and $f(x)$ we can put them into the derivative formula.

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{(1 - x^2 - 2xh - h^2) - (1 - x^2)}{h}$ Here, $f(x+h) = 1 - x^2 - 2xh - h^2$ and $f(x) = 1 - x^2$.

$f'(x) = \lim_{h\to 0} \frac{1 - x^2 - 2xh - h^2 - 1 + x^2}{h}$ We will distribute the minus sign and then simplify the numerator.

$f'(x) = \lim_{h\to 0} \frac{-2xh - h^2}{h}$ In order to do the limit we must factor out the h and cancel it out.

$f'(x) = \lim_{h\to 0} \frac{h(-2x - h)}{h} = \lim_{h\to 0} -2 - h$. Now that we canceled, we can replace the h with a 0.

$f'(x) = \lim_{h\to 0} -2x - h = -2 - 0 = -2x$ So we write $f'(x) = -2x$. This is an expression that will allow us to find the slope of f at any value of x.

Now let's use our answer to find $f'(-3)$. We will put in a -3 into the equation -2x to get: $f'(-3) = -2(-3) = 6$.
We do the same for the others: $f'(0) = -2(0) = 0$. $f'\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right) = -1$

EXAMPLE: Find the derivative of $f(x) = -5$ by the limit process. Then use your answer to find: $f'(-100)$

This one is a lot easier because there are no variables. What is $f(x+h)$? Since there are no variables to plug $x+h$ into then $f(x+h) = -5$. So in the derivative formula we'll put in a -5 for both $f(x+h)$ and $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = \lim_{h \to 0} \frac{-5-(-5)}{h} = \lim_{h \to 0} \frac{0}{h}$ Now we will get a zero on top causing the whole thing to be zero.

$f'(x) = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$ So $f'(x) = 0$. This will happen whenever we are taking the slope of a single number with no variables. Let's think about why this happens. $f(x) = -5$ is a horizontal line crossing the y-axis at -5. Horizontal lines are not increasing or decreasing, so the slope should be zero.

When we find $f'(-100)$, the answer would be zero since at any x value the slope is always zero.

EXAMPLE: Find the derivative of $f(x) = \sqrt{x} + 6$ by the limit process.

This one is going to be done a little different. First we need to find $f(x+h)$.

$f(x+h) = \sqrt{x+h} + 6$ We just put in a $x+h$ for x in our original equation for f. We are ready to substitute this information into our derivative formula.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} + 6 - (\sqrt{x} + 6)}{h}$ Here, $f(x+h) = \sqrt{x+h} + 6$ and $f(x) = \sqrt{x} + 6$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} + 6 - \sqrt{x} - 6}{h}$ Make sure you remember to distribute the negative and cancel the 6 & -6.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ Okay so now we need to multiply top and bottom by the conjugate of the top.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$ This was the same technique used in the last chapter.

$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h(\sqrt{x+h} + \sqrt{x})}$ We will apply the difference of squares when multiplying on top.

$f'(x) = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$ So not the square roots cancelled, so we now cancel the x and $-x$.

$f'(x) = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$ Now we can cancel the h from the top and bottom.

$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$ Now we can put in a 0 for h, simplify, and we are done.

$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. So our final answer is $f'(x) = \frac{1}{2\sqrt{x}}$.

EXAMPLE: Find the derivative of $f(x)=\dfrac{4}{\sqrt{x}}$ by the limit process.

First we will find $f(x+h)$. This will be $f(x+h)=\dfrac{4}{\sqrt{x+h}}$. I will leave it in this form. Now we will put in a $\dfrac{4}{\sqrt{x+h}}$ for $f(x+h)$ and a $\dfrac{4}{\sqrt{x}}$ for $f(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4}{\sqrt{x+h}}-\dfrac{4}{\sqrt{x}}}{h}$$ So now we need common denominators in the numerator.

$$f'(x)=\lim_{h\to 0}\frac{\left(\dfrac{\sqrt{x}}{\sqrt{x}}\right)\dfrac{4}{\sqrt{x+h}}-\dfrac{4}{\sqrt{x}}\left(\dfrac{\sqrt{x+h}}{\sqrt{x+h}}\right)}{h}$$ Multiply each by a power of one.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4\sqrt{x}}{\sqrt{x}\left(\sqrt{x+h}\right)}-\dfrac{4\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}}{h}$$ Now combine to get a single fraction on top.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{4\sqrt{x}-4\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}}{\dfrac{h}{1}}$$ Now let's clear the double fractions

$$f'(x)=\lim_{h\to 0}\frac{4\sqrt{x}-4\sqrt{x+h}}{h\sqrt{x}\left(\sqrt{x+h}\right)}$$ Now we need to multiply by the conjugate again.

$$f'(x)=\lim_{h\to 0}\frac{4\sqrt{x}-4\sqrt{x+h}}{h\sqrt{x}\left(\sqrt{x+h}\right)}\cdot\frac{4\sqrt{x}+4\sqrt{x+h}}{4\sqrt{x}+4\sqrt{x+h}}$$ We will use the difference of squares again.

$$f'(x)=\lim_{h\to 0}\frac{\left(4\sqrt{x}\right)^2-\left(4\sqrt{x+h}\right)^2}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now simplify the top.

$$f'(x)=\lim_{h\to 0}\frac{16x-16(x+h)}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now distribute the negative and cancel the 16x terms.

$$f'(x)=\lim_{h\to 0}\frac{-16h}{h\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ Now cancel the *h* from the top and bottom.

$$f'(x)=\lim_{h\to 0}\frac{-16}{\sqrt{x}\left(\sqrt{x+h}\right)\left(4\sqrt{x}+4\sqrt{x+h}\right)}$$ So now we put in a 0 for *h*.

$$f'(x)=\frac{-16}{\sqrt{x}\left(\sqrt{x+0}\right)\left(4\sqrt{x}+4\sqrt{x+0}\right)}=\frac{-16}{\sqrt{x}\sqrt{x}(4\sqrt{x}+4\sqrt{x})}=\frac{-16}{x\left(8\sqrt{x}\right)}=\frac{-16}{8x\sqrt{x}}=\frac{-2}{x\sqrt{x}}$$ So $f'(x)=\dfrac{-2}{x\sqrt{x}}$.

EXAMPLE: Find the derivative of $f(x)=\dfrac{1}{x^2}$ by the limit process.

First we will find $f(x+h)$. This will be $f(x+h)=\dfrac{1}{(x+h)^2}$. I will leave it in this form. Now we will put in a $\dfrac{1}{(x+h)^2}$ for $f(x+\Delta x)$ and a $\dfrac{1}{x^2}$ for $f(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$$ So now we need common denominators in the numerator.

$$f'(x)=\lim_{h\to 0}\frac{\left(\dfrac{x^2}{x^2}\right)\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\left(\dfrac{(x+h)^2}{(x+h)^2}\right)}{h}$$ We multiply each fraction by a power of one.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{x^2}{x^2(x+h)^2}-\dfrac{(x+h)^2}{x^2(x+h)^2}}{h}$$ Now we can combine the two top fractions into one.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}}{\dfrac{h}{1}}$$ We need to get rid of these double fractions. Multiply by the reciprocal.

$$f'(x)=\lim_{h\to 0}\frac{x^2-(x+h)^2}{(h)x^2(x+h)^2}$$ We need to simplify the numerator by multiplying.

$$f'(x)=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$$ Now distribute the negative and cancel the x^2.

$$f'(x)=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}$$ Now we factor out an *h* from the numerator.

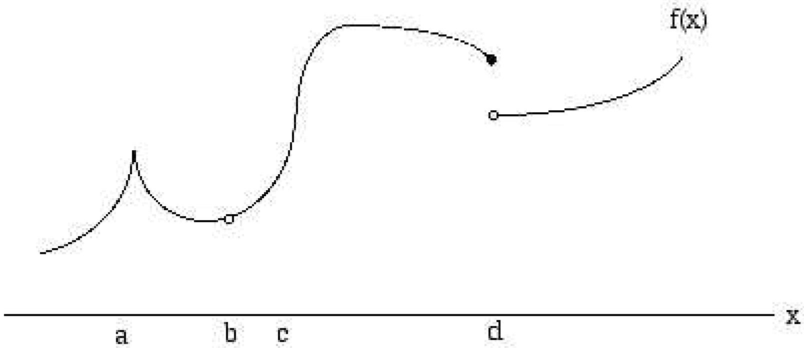
$$f'(x)=\lim_{h\to 0}\frac{h(-2x-h)}{hx^2(x+h)^2}$$ We can cancel out the *h* from the top and bottom.

$$f'(x)=\lim_{h\to 0}\frac{(-2x-h)}{x^2(x+h)^2}$$ So now we are finally about to put in a 0 for h.

$$f'(x)=\lim_{h\to 0}\frac{(-2x-h)}{x^2(x+h)^2}=\frac{(-2x-0)}{x^2(x+0)^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}.$$ So we finally have our answer: $f'(x)=\dfrac{-2}{x^3}$.

When does a derivative *not* have a derivative at a point?

Because we find derivatives using limits, what if the limit at a certain point does not exist? This means there is no derivative at that point. In the graph below, there are four points where the derivative does not exist.



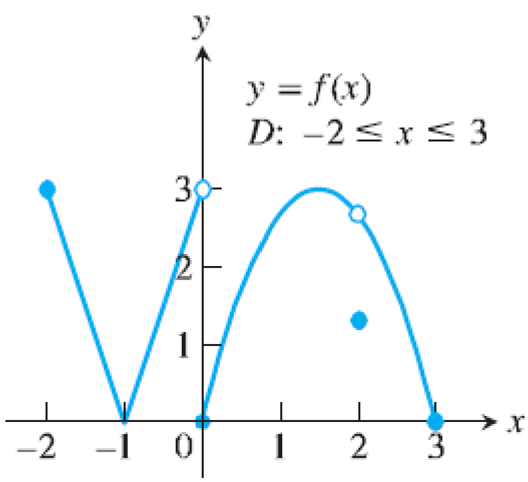
At point a: There is a cusp (or corner) so the derivative does not exist at a.

At point b: There is a hole in the graph, so the derivative does not exist at b.

At point c: There is a vertical tangent here, so the derivative does not exist at c.

At point d: There is a discontinuity at d (gap in graph), so the derivative does not exist at d.

EXAMPLE: The figure below shows a graph of a function over the closed interval [-2, 3]. At what x-value(s) does the graph appear to be: a.) continuous but not differentiable? b.) neither continuous nor differentiable?



a.) At x = -1 the graph would be continuous (no break) however, the derivative would not exist here because there is a corner.

b.) At $x = 0$ and $x = 2$ the graph is not continuous, so because of this, derivative does not exist here either.