3.3 Differentiation Rules

The derivative has several different types of notation:

Notation	Meaning
$f'(x)$	Derivative of $f(x)$
$\frac{dy}{dx}$	Derivative of y with respect to x.
y'	Derivative of y
$\frac{d}{dx}[f(x)]$	Derivative of f with respect to x.

Constant Rule

$\frac{d}{dx}[c]=0$ This means the derivative of any number is zero. For example, suppose we had $y=9$ and the question asked us to find y'. Then our answer would automatically be zero because y = 9 is a horizontal line. The slope of a horizontal line is always 0.

EXAMPLE: Find $f'(x)$ if $f(x)=4\cdot\pi\cdot e^2$.

Since $e=2.71...$ and $\pi=3.14...$ then this whole equation is a constant, so $f'(x)=0$.

Can you see a pattern of the derivative with the following table?

Original function	Derivative
$y=x^2$	$y'=2x$
$y=x^3$	$y'=3x^2$
$y=x^5$	$y'=5x^4$

It seems like the power comes down and the exponent is reduced by one. We can use this information to derive a formula for the derivative for $y=x^n$. This will be called the power rule.

Power Rule

If n is any real number, then $\frac{d}{dx}x^n=nx^{n-1}$ for all x where powers of x^n and x^{n-1} are defined.

Other Derivative Rules

$$\frac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$$
$$\frac{d}{dx}[f(x)-g(x)]=f'(x)-g'(x)$$
$$\frac{d}{dx}[c\cdot f(x)]=c\cdot f'(x)$$

EXAMPLE: If $y = x^{12}$, find y'.

We will use the power rule formula. Here n = 12. $y' = 12x^{12-1}$ so $y' = 12x^{11}$.
EXAMPLE: If $y = 3x^4$, find y'.

We can write y as $y = 3 \cdot x^4$. Now we use the power rule: $y' = 3 \cdot 4x^3$, so $y' = 12x^3$

EXAMPLE: If $g(x) = \frac{3}{2}x^6 - x + 3$, find $g'(x)$.

First we will rewrite this as $g(x) = \frac{3}{2} \cdot x^6 - 1x^1 + 3x^0$. For this one we can apply the power rule to each term separately: $g'(x) = \frac{3}{2} \cdot 6x^5 - 1x^0 + 0$. After simplifying we get: $g'(x) = 9x^5 - 1$.

EXAMPLE: If $y = 3x(6x - 5x^2)$, find y'.

I recommend distributing first. You will get: $y = 18x^2 - 15x^3$. Now apply the power rule. You will get: $y' = 36x - 45x^2$. You can also write it as $y' = 9x(4 - 5x)$

EXAMPLE: If $f(x) = \sqrt[4]{x}$, find $f'(x)$.

This one can be rewritten with fractional exponents: $f(x) = x^{\frac{1}{4}}$. We still can apply the power rule here: $f(x) = \frac{1}{4}x^{\frac{1}{4}-1}$, so after simplifying you get $f(x) = \frac{1}{4}x^{-\frac{3}{4}}$. You do not want this as a negative exponent, so place x in the denominator. You will get $f(x) = \frac{1}{4x^{\frac{3}{4}}}$.

EXAMPLE: If $f(x) = \frac{2}{x^3}$, find $f'(x)$.

If there is a variable in the denominator, write this as a negative exponent. You will get $f(x) = 2x^{-3}$. Then apply the power rule. You will get $f'(x) = -6x^{-4}$. Now write it without negative exponents and you will have $f'(x) = -\frac{6}{x^4}$.
EXAMPLE: If $f(x) = x + \frac{1}{x}$, find $f'(x)$.

We need to first rewrite this as a negative exponent. You will get $f(x) = x + x^{-1}$. Apply the power rule. You will get $f'(x) = 1 - x^{-2}$. Now rewrite without negative exponents: $f'(x) = 1 - \frac{1}{x^2}$.

EXAMPLE: If $f(x)=\dfrac{2x^2-3x+1}{x}$, find $f'(x)$.

I would suggest dividing each term by x and then simplifying: $f(x)=\dfrac{2x^2}{x}-\dfrac{3x}{x}+\dfrac{1}{x}$ so $f(x)=2x-3+x^{-1}$.

Now apply the power rule: $f'(x)=2-x^{-2}$. Finally rewrite with positive exponents: $f'(x)=2-\dfrac{1}{x^2}$.

EXAMPLE: If $y=\sqrt{x}-\dfrac{5}{\sqrt{x}}+\sqrt{2}$, find y'.

First write change the radicals into fractional exponents: $y=x^{\frac{1}{2}}-5x^{-\frac{1}{2}}+\sqrt{2}$. I did not change the last square root of 2 into a fractional exponent because when I take the derivative this term will turn to zero. Now apply the product rule: $y'=\dfrac{1}{2}x^{-\frac{1}{2}}+\dfrac{5}{2}x^{-\frac{3}{2}}$. Lastly rewrite with positive exponents: $y'=\dfrac{1}{2x^{\frac{1}{2}}}+\dfrac{5}{2x^{\frac{3}{2}}}$. You may leave it with fraction exponents unless the question specifically tells you not to. Suppose this question did want you to write it with radicals. Then you would have: $y'=\dfrac{1}{2\sqrt{x}}+\dfrac{5}{2\sqrt{x^3}}$, which is also $y'=\dfrac{1}{2\sqrt{x}}+\dfrac{5}{2x\sqrt{x}}$.

With common denominators it can be written as $y'=\dfrac{x+5}{2x\sqrt{x}}$.

EXAMPLE: Determine the point(s) at which $y=x^2+1$ has a horizontal tangent line.

A horizontal tangent line means that the slope is zero. So we must take the derivative and set it equal to zero.

$y'=2x$	We have our derivative. Now we must set it equal to zero.
$0=2x$	Now we solve for x.
$x=0$	So when x = 0, y = 1, so the point is (0, 1).

EXAMPLE: Determine the point(s) at which $y=x^3-27x$ has a horizontal tangent line.

$y'=3x^2-27$	We have our derivative. Now we must set it equal to zero.
$0=3x^2-27$	Now we solve for x.
$27=3x^2$	
$9=x^2$	
$x=\pm 3$	Put each of these into the original equation for x and we get (3, -54) and (-3, 54).

We unfortunately can't use the product rule on everything. We will look at ways to find derivatives of things multiplied together and divided.

Product Rule

$$\frac{d}{dx}[f \cdot g] = f \cdot g' + g \cdot f'$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g \cdot f' - f \cdot g'}{g^2}$$

EXAMPLE: Use the product rule to find $H'(x)$ if $H(x) = \underset{f}{(6x+5)}\underset{g}{(x^3-2)}$.

In our problem, I labeled f and g. Now we apply the product rule formula. I have labeled each part.

$H'(x) = \underset{f}{(6x+5)}\underset{g'}{(3x^2)} + \underset{g}{(x^3-2)}\underset{f'}{(6)}$ Now we just simplify.

$H'(x) = 18x^3 + 15x^2 + 6x^3 - 12$

$H'(x) = 24x^3 + 15x^2 - 12$

EXAMPLE: Use the quotient rule to find $H'(x)$ if $H(x) = \dfrac{x^2+2}{2x-7}$.

Here the numerator is always f and the denominator is always g. Now we use the quotient rule:

$H'(x) = \dfrac{\underset{g}{(2x-7)}\underset{f'}{(2x)} - \underset{f}{(x^2+2)}\underset{g'}{(2)}}{\underset{g^2}{(2x-7)^2}}$ Now we distribute and simplify.

$H'(x) = \dfrac{4x^2 - 14x - 2x^2 - 4}{(2x-7)^2}$ Now just simplify to get our answer: $H'(x) = \dfrac{2x^2 - 14x - 4}{(2x-7)^2}$

EXAMPLE: Use the product rule to find $M'(x)$ if $M(x) = \underset{f}{\sqrt{x}}\underset{g}{(4-x^2)}$. Write answer as a single fraction.

Once again f and g are labeled. We need to use the product rule again.

$M'(x) = \underset{f}{\sqrt{x}}\underset{g'}{(-2x)} + \underset{g}{(4-x^2)}\underset{f'}{\frac{1}{2}x^{-\frac{1}{2}}}$ Now get rid of the negative exponent and rewrite with radicals.

$M'(x) = -2x\sqrt{x} + \frac{4 - x^2}{2\sqrt{x}}$ We need common denominators to write this as a single fraction.

$M'(x) = \frac{-2x\sqrt{x}}{1} \cdot \left(\frac{2\sqrt{x}}{2\sqrt{x}}\right) + \frac{4 - x^2}{2\sqrt{x}}$ We need to multiply across the top and across the bottom.

$M'(x) = \frac{-4x^2}{2\sqrt{x}} + \frac{4 - x^2}{2\sqrt{x}}$ Now combine as a single fraction.

$M'(x) = \frac{4 - 5x^2}{2\sqrt{x}}$

EXAMPLE: Use the product rule to find $M'(x)$ if $M(x) = \underset{f}{(x^2 + 1)}\underset{g}{(x + 5 + 1/x)}$.

$M'(x) = \underset{f}{(x^2 + 1)}\underset{g'}{\left(1 - \frac{1}{x^2}\right)} + \underset{g}{\left(x + 5 + \frac{1}{x}\right)}\underset{f'}{(2x)}$ Now multiply this together.

$M'(x) = x^2 - 1 + 1 - \frac{1}{x^2} + 2x^2 + 10x + 2$ We simplify and add like terms.

$M'(x) = 3x^2 + 10x + 2 - \frac{1}{x^2}$ We can leave our answer like this.

EXAMPLE: Use the quotient rule to find $H'(x)$ if $H(x) = \frac{x}{\sqrt{x} - 1}$.

We will use the quotient rule again. Remember the numerator is always f and the denominator is always g.

$H'(x) = \frac{(\sqrt{x} - 1)(1) - x \cdot \frac{1}{2} x^{-\frac{1}{2}}}{(\sqrt{x} - 1)^2}$ Now we will simplify. The x's cancel this way: $x \cdot x^{-\frac{1}{2}} = x^{\frac{1}{2}}$.

$H'(x) = \frac{\sqrt{x} - 1 - \frac{1}{2} x^{\frac{1}{2}}}{(\sqrt{x} - 1)^2}$ Now we will multiply the top and bottom by 2 to clear the fraction.

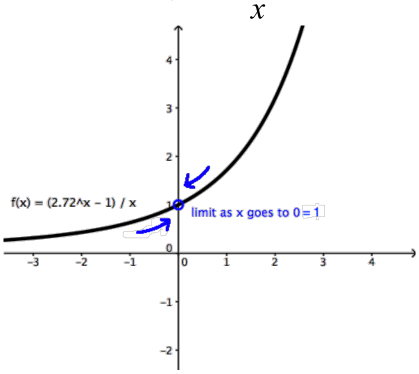
$H'(x) = \frac{2}{2} \cdot \frac{\sqrt{x} - 1 - \frac{1}{2} x^{\frac{1}{2}}}{(\sqrt{x} - 1)^2}$

$H'(x) = \frac{2\sqrt{x} - 2 - \sqrt{x}}{2(\sqrt{x} - 1)^2}$ I multiplied by 2 and then changed the fractional power back into a radical.

$H'(x) = \frac{\sqrt{x} - 2}{2(\sqrt{x} - 1)^2}$ I added the like terms on top and this is as far as we can go.

Derivative of e^x

We want to derive the derivative of e^x. So far the only way we can do this is using the limit process. First we will start with the definition: $\frac{d}{dx}\left[e^x\right]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. In this case, $f(x)=e^x$, and $f(x+h)=e^{x+h}$. Let's substitute this into the limit definition: $\frac{d}{dx}\left[e^x\right]=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$. We can rewrite this as: $\frac{d}{dx}\left[e^x\right]=\lim_{h\to 0}\frac{e^x e^h-e^x}{h}$. Factor out e^x: $\frac{d}{dx}\left[e^x\right]=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}$. Then we can rewrite as: $\frac{d}{dx}\left[e^x\right]=e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}$. So now let's look at the graph of $y=\frac{e^x-1}{x}$ (on next page).



So we see here from the graph that $\lim_{x\to 0}\frac{e^x-1}{x}=1$. Therefore $\lim_{h\to 0}\frac{e^h-1}{h}=1$. So in $\frac{d}{dx}\left[e^x\right]=e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}$, we can substitute 1 for $\lim_{h\to 0}\frac{e^h-1}{h}$. Then we get: $\frac{d}{dx}\left[e^x\right]=e^x(1)$.

Therefore, $\frac{d}{dx}\left[e^x\right]=e^x$. So the derivative of e^x is itself!

EXAMPLE: Given: $y=e^{-x}$ find y'.

We have just seen the derivative of e^x, however e^{-x} often comes up in problems, so let's first find this derivative. First let's rewrite our equation as $y=\frac{1}{e^x}$. So now we need to use the quotient rule: $y'=\frac{e^x(0)-1\left(e^x\right)}{\left(e^x\right)^2}$. Now let's simplify: $y'=\frac{-e^x}{e^{2x}}=-\frac{1}{e^x}=-e^{-x}$. So now we can say $\frac{d}{dx}\left[e^{-x}\right]=-e^{-x}$.

EXAMPLE: Given: $y=\frac{e^x-e^{-x}}{2}$ find y'.

I will first divide each term by 2: $y=\frac{1}{2}e^x-\frac{1}{2}e^{-x}$. Now we can take the derivative of each term separately: $y'=\frac{1}{2}e^x-\frac{1}{2}\left(-e^{-x}\right)$. Simplifying we get: $y'=\frac{1}{2}e^x+\frac{1}{2}e^{-x}$. Notice I used $\frac{d}{dx}\left[e^{-x}\right]=-e^{-x}$ from above.

EXAMPLE: Given: $y=x^2e^x$ find y'.

For this one we need to use the product rule. The derivative of e^x is e^x as we proved earlier:
$y'=x^2e^x+e^x(2x)$. This can be simplified: $y'=x^2e^x+2xe^x$. You could also factor this one: $y'=xe^x(x+2)$.

EXAMPLE: Given: $y = (9x^2 - 6x + 2)e^x$ find y'.

For this one we also need to use the product rule: $y' = (9x^2 - 6x + 2)e^x + e^x(18x - 6)$. We can factor out the e^x from both terms: $y' = e^x(9x^2 - 6x + 2 + 18x - 6)$. After simplifying you will get: $y' = e^x(9x^2 + 12x - 4)$.

EXAMPLE: Given: $y = \sqrt[11]{x^3} - e^3 + x^e$ find y'.

First let's rewrite this one: $y = x^{\frac{3}{11}} - e^3 + x^e$. To find the derivative of the first term we will use the power rule. For the second term, the derivative is zero because e^3 is a constant. For the last term we can still use the power rule: $y' = \frac{3}{11}x^{-\frac{8}{11}} - 0 + ex^{e-1}$. This simplifies to: $y' = \frac{3}{11x^{\frac{8}{11}}} + ex^{e-1}$. Nothing more we can do here.

EXAMPLE: Given: $y = \frac{4e^x}{2x^5 - 3e^x}$ find y'.

This problem requires the quotient rule: $y = \frac{(2x^5 - 3e^x)4e^x - 4e^x(10x^4 - 3e^x)}{(2x^5 - 3e^x)^2}$. We can factor the out the $4e^x$ from the top: $y' = \frac{4e^x(2x^5 - 3e^x - (10x^4 - 3e^x))}{(2x^5 - 3e^x)^2}$. Distribute the negative: $y' = \frac{4e^x(2x^5 - 3e^x - 10x^4 + 3e^x)}{(2x^5 - 3e^x)^2}$.
Now simplify: $y' = \frac{4e^x(2x^5 - 10x^4)}{(2x^5 - 3e^x)^2}$. Finally we can do one more factoring step: $y' = \frac{8x^4e^x(x-5)}{(2x^5 - 3e^x)^2}$.

Higher Order Derivatives

$f(x)$	This is our original function
$f'(x)$	First derivative of f
$f''(x)$	Second derivative of f (derivative of $f'(x)$)
$f'''(x)$	Third derivative of f (derivative of $f''(x)$)
$f^{(n)}(x)$	The nth derivative of f (derivative of $f^{(n-1)}(x)$)

A function has derivatives of all orders once the derivative reaches zero.

EXAMPLE: Let $f(x) = 4x^3 + 5x^2 + 3x + 1$. Find the derivatives of all orders.

We will use the power rule for this.

$f'(x) = 12x^2 + 10x + 3$	In order to find $f''(x)$, take the derivative of $f'(x)$ using the power rule.
$f''(x) = 24x + 10$	Now we will take the derivative of $f''(x)$ to get $f'''(x)$.
$f'''(x) = 24$	
$f^{(4)}(x) = 0$	The derivative is zero, and all subsequent derivatives are zero, so we've found the derivatives of all orders.

EXAMPLE: Let $f(x) = \frac{x^3 + 2x^2 - 1}{x}$. Find $f'''(x)$.

You might think that we need to use the quotient rule for this one, but we don't need to. We can simplify this by dividing each term by x: $f(x) = \frac{x^3}{x} + \frac{2x^2}{x} - \frac{1}{x}$. This simplifies to $f(x) = x^2 + 2x - x^{-1}$

$f(x) = x^2 + 2x - x^{-1}$	
$f'(x) = 2x + 2 + x^{-2}$	Take the first derivative using the power rule.
$f''(x) = 2 - 2x^{-3}$	In order to find $f''(x)$, take the derivative of $f'(x)$ using the power rule.
$f'''(x) = 6x^{-4}$	Now we will take the derivative of $f''(x)$ to get $f'''(x)$.
$f'''(x) = \frac{6}{x^4}$	

EXAMPLE: Given: $y = \left(\frac{x^3 - 2}{5x}\right)\left(\frac{x^2 + 5}{x^3}\right)$ find y' and y''.

Instead of applying the quotient rule twice with the product rule, let's first multiply this out: $y = \frac{x^5 + 5x^3 - 2x^2 - 10}{5x^4}$. Now we can break up this fraction: $y = \frac{1}{5}x + x^{-1} - \frac{2}{5}x^{-2} - 2x^{-4}$. To find the first derivative we will use the power rule: $y' = \frac{1}{5} - x^{-2} + \frac{4}{5}x^{-3} + 8x^{-5}$. We will take the derivative of this to find the second derivative: $y'' = 2x^{-3} - \frac{12}{5}x^{-4} - 40x^{-6}$. So for our final answers we will write: $y' = \frac{1}{5} - \frac{1}{x^2} + \frac{4}{5x^3} + \frac{8}{x^5}$ and $y'' = \frac{2}{x^3} - \frac{12}{5x^4} - \frac{40}{x^6}$.

EXAMPLE: Given: $y = 4x^3e^{-x}$ find y' and y''.

First we will find the first derivative using the product rule: $y' = 4x^3\left(-e^{-x}\right) + e^{-x}(12x^2)$. This simplifies to: $y' = -4x^3e^{-x} + 12x^2e^{-x}$. We want to find the derivative of this answer to get the second derivative. Each term requires use of the product rule: $y'' = -4x^3\left(-e^{-x}\right) + e^{-x}(-12x^2) + 12x^2\left(-e^{-x}\right) + e^{-x}(24x)$. Now simplify: $y'' = 4x^3e^{-x} - 24x^2e^{-x} + 24xe^{-x}$. Then we can factor: $y'' = 4xe^{-x}\left(x^2 - 6x + 6\right)$. Also $y' = 4x^2e^{-x}(3 - x)$.